

Section 16.8 Triple Integrals in Spherical Coordinates

“Integrating Functions in Different Coordinate Systems”

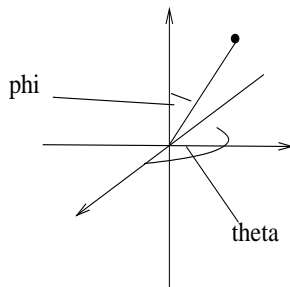
In the previous section, we used cylindrical coordinates to help evaluate triple integrals. In this section we introduce a second coordinate system, called spherical coordinates, to make integrals over spherical regions easier.

1. SPHERICAL COORDINATES

The second set of coordinates we consider are a little more complicated. They are particularly useful when describing regions or surfaces which are similar to a sphere.

Definition 1.1. We define the spherical coordinates $(\rho, \vartheta, \varphi)$ of a point P in space as follows:

- ρ is the distance of P from the origin (so $\rho \geq 0$).
- ϑ is the same angle as in cylindrical coordinates (the angle made from the positive x -axis, so $0 \leq \vartheta < 2\pi$).
- φ is the angle between the positive z -axis and the line segment connecting P to the origin (so $0 \leq \varphi \leq \pi$).



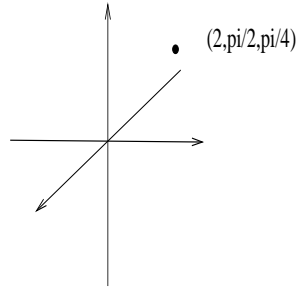
Conversion between these coordinate systems is a little more complicated. It is done using the following formulas.

Result 1.2. (i) The rectangular coordinates of the point $(\rho, \vartheta, \varphi)$ in 3 space are $x = \rho \sin(\varphi) \cos(\vartheta)$, $y = \rho \sin(\varphi) \sin(\vartheta)$ and $z = \rho \cos(\varphi)$.

(ii) The spherical coordinates of the point (x, y, z) can be found by solving the equation $\rho^2 = x^2 + y^2 + z^2$, and then using the equations given above.

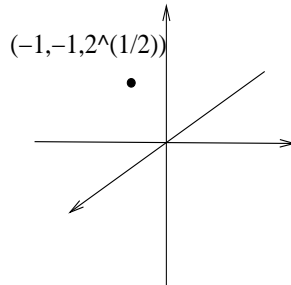
Remember that when converting from rectangular to spherical, you need to be very careful with the angles ϑ and φ (WHY?). We illustrate with a few examples.

Example 1.3. Plot the point with spherical coordinates $(2, \pi/2, \pi/4)$ and convert to rectangular coordinates.



Converting, we have $x = 0$, $y = \sqrt{2}$ and $z = \sqrt{2}$, so $(0, \sqrt{2}, \sqrt{2})$.

Example 1.4. Plot the point with rectangular coordinates $(-1, -1, \sqrt{2})$ and convert to spherical coordinates.



Converting, we have $\rho^2 = 1 + 1 + 2 = 4$, so $\rho = 2$. Then we have $\sqrt{2} = 2 \sin(\varphi)$, so $\sin(\varphi) = \sqrt{2}/2$, or $\varphi = \pi/4$. Finally, $-1 = 2 \sin(\varphi) \cos(\vartheta)$ and $-1 = 2 \sin(\varphi) \sin(\vartheta)$ giving $\sin(\varphi) = \cos(\varphi) = -\sqrt{2}/2$, or $\vartheta = 5\pi/4$. So the coordinates are $(2, 5\pi/4, \pi/4)$.

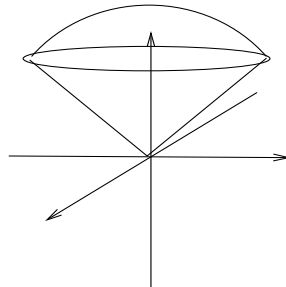
Example 1.5. Write the equation $z = x^2 + y^2$ in spherical coordinates.

We know $x^2 + y^2 + z^2 = \rho^2$, so we can replace the right hand side by $\rho^2 - z^2$ giving $z = \rho^2 - z^2$ or $z^2 + z = \rho^2$. Since $z = \rho \cos(\varphi)$, we have $\rho^2 \cos^2(\varphi) + \rho \cos(\varphi) = \rho^2$, so simplifying, we get

$$\rho \cos^2(\varphi) + \cos(\varphi) = \rho.$$

Example 1.6. Sketch the solid bounded by $0 \leq \varphi \leq \pi/4$, and $\rho \leq 1$.

We can describe this solid in words - the angle from the z -axis is bounded from 0 to $\pi/4$ and the distance from the origin is less than or equal to 1, and there are no restrictions on ϑ . This suggests a shape like an ice-cream cone as illustrated below:



2. TRIPLE INTEGRALS IN SPHERICAL COORDINATES

Integrals in spherical coordinates are a little more complicated to calculate than regular or cylindrical integrals - and integrals in spherical coordinates over general regions are usually very difficult. For this reason, we usually only consider spherical regions over “spherical boxes” (portions of wedges). To calculate such an integral, we use the following formula:

Result 2.1. If R is a spherical box, so $a \leq \rho \leq b$, $\alpha \leq \vartheta \leq \beta$ and $c \leq \varphi \leq d$, then

$$\begin{aligned} & \int \int \int_R f(x, y, z) dV \\ &= \int_c^d \int_\alpha^\beta \int_a^b \left(f(\rho \sin(\varphi) \cos(\vartheta), \rho \sin(\varphi) \sin(\vartheta), \rho \cos(\varphi)) \right) \rho^2 \sin(\varphi) d\rho d\vartheta d\varphi \end{aligned}$$

Thus to evaluate an integral in spherical coordinates, we do the following:

- (i) Convert the function $f(x, y, z)$ into a spherical function.
- (ii) Change the limits of the integral and include the “ $\rho^2 \sin(\varphi)$ ” in the integral.
- (iii) Evaluate.

As with cylindrical coordinates, we could evaluate more general spherical regions where the limits are functions of the variables, though often this will result in a more complicated integral. We illustrate with a couple of examples.

Example 2.2. (i) Use spherical coordinates to evaluate

$$\int \int \int_R 3e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV$$

where R is the region inside the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

In spherical coordinates, the region is $0 \leq \varphi \leq \pi/2$, $0 \leq \vartheta \leq \pi/2$ and $0 \leq \rho \leq 3$. Thus we need to evaluate the following:

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 3e^{\rho^3} \rho^2 \sin(\varphi) d\rho d\vartheta d\varphi = \int_0^{\pi/2} \int_0^{\pi/2} e^{\rho^3} \sin(\varphi) \Big|_0^3 d\vartheta d\varphi \\ &= \int_0^{\pi/2} \int_0^{\pi/2} (e^{27} - 1) \sin(\varphi) d\vartheta d\varphi = \int_0^{\pi/2} \frac{\pi(e^{27} - 1)}{2} \sin(\varphi) d\varphi \\ &= \left[-\frac{\pi(e^{27} - 1)}{2} \cos(\varphi) \right]_0^{\pi/2} = \frac{\pi(e^{27} - 1)}{2} \end{aligned}$$

- (ii) Set up, but do not evaluate, a spherical integral of the function $f(x, y, z)$ over the solid cone $\sqrt{x^2 + y^2} \leq z \leq 2$.

Here we have $0 \leq \vartheta \leq 2\pi$. To find the limits on φ , we observe that the angle the cone makes with the z -axis is $\pi/4$, so we have $0 \leq \varphi \leq \pi/4$. Observe that the upper ϱ -limits depend upon φ , but not ϑ . To evaluate these limits, we observe that $z \leq 2$. We know $\varrho^2 = x^2 + y^2 + z^2$, so converting to spherical and using $z = 2$, we have

$$\varrho^2 = \varrho^2 \sin^2(\varphi) \cos^2(\vartheta) + \varrho^2 \sin^2(\varphi) \sin^2(\vartheta) + 4 = \varrho^2 \sin^2(\varphi) + 4$$

Solving for ϱ^2 , we have

$$\varrho^2 = \frac{4}{1 - \sin^2(\varphi)} = \frac{4}{\cos^2(\varphi)}$$

Since $\varrho \geq 0$, we have

$$\varrho = \frac{2}{\cos(\varphi)}$$

Thus the integral will be

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\frac{2}{\cos(\vartheta)}} f(\varrho, \vartheta, \varrho) \varrho^2 \sin(\varphi) d\varrho d\varphi d\vartheta$$