Section 17.2
Line Integrals

"Integrating Vector Fields and Functions along a Curve"

In this section we consider the problem of integrating functions, both scalar and vector (vector fields) along a curve C in the plane. We want the definition to generalize the ideas of integration we have already developed in Calculus I and II. We start with the scalar problem.

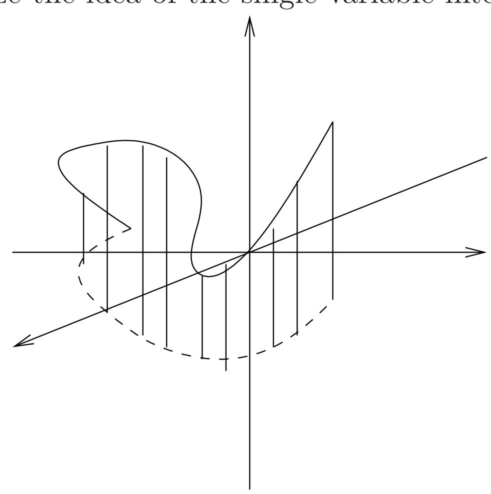
1. Line Integrals of a Scalar Function with respect to Distance, x and y

We want to define the integral of a continuous function $f(x, y)$ of two variables over a general smooth curve C in the plane. We do this as follows:

- (i) Suppose C is some curve in space parameterized by the functions $(x(t), y(t))$ with $a \leqslant t \leqslant b$, or equivalently with vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.
- (ii) Suppose $f(x, y)$ is a function which is continuous on C.
- (iii) Break up the interval $[a, b]$ into n equal sized intervals of lengths $\Delta s = (b - a)/n$.
- (iv) Fix a point $(\bar{x}_i, \bar{y}_i)$ in each interval.
- (v) We can construct the Riemann sum

$$\sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta s$$

- (vi) Observe that this Riemann sum approximates the area bounded between the function $f(x, y)$ and the curve C weighted by the xy-plane (see illustration below). In particular, this really does generalize the idea of the single variable integral.



We are now ready to formally define a line integral with respect to distance.

Definition 1.1. If $f(x, y)$ is defined on a smooth curve C given by the equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, then the integral of f along C is defined by

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta s$$

provided this limit exists.

Observe that the integral is with respect to s and the function itself is with respect to x and y. This means we cannot directly integrate it. However, we can integrate using the following method:

(i) Rewrite $f(x, y)$ in terms of t by substituting $x = x(t)$ and $y = y(t)$.

(ii) Replace ds in the integral by

$$||\vec{r}'(t)|| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

(substitution).

(iii) We can now integrate with respect to t with limits $a \leqslant t \leqslant b$.

Summarizing, we evaluate line integrals using the following:

Result 1.2.

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

$$= \int_a^b f(x(t), y(t))||\vec{r}'(t)||dt$$

By the definition of the line integral, we are integrating with respect to the arc length s. There are two other line integrals we shall be interested - line integrals with respect to x and y. Specifically, we can modify the formula, so instead of integrating with respect to s, we integrate with respect to x or y. In this case, since $x = x(t)$ and $y = y(t)$, we would have either $dx = x'(t)dt$ or $dy = y'(t)dt$, giving the following formulas for such integrals:

Result 1.3.

$$\int_C f(x, y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$

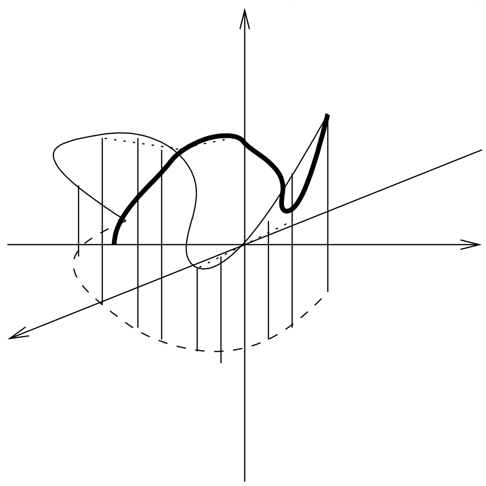
$$\int_C f(x, y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

Often, line integrals are expressed as integrals in terms of integrals of x and y, so they are often abbreviated to the following notation:

Result 1.4.

$$\int_C (P(x,y)dx + Q(x,y)dy) = \int_a^b P(x(t),y(t))x'(t)dt + \int_a^b Q(x(t),y(t))y'(t)dt$$

The geometric interpretation is a little more difficult than the line integral with respect to x and y, but roughly it can be interpreted as the weighted area under the projections of the curve $z = f(x,y)$ in the xz-plane or the yz-plane (see illustration) (where we count area as negative if the curve C moves in the negative y or negative x-direction).



We illustrate with some examples.

Example 1.5. (*i*) Calculate $\int_C (xy + \ln(x))dy + xdx$ where C is the arc of the parabola $y = x^2$ from $(1,1)$ to $(3,9)$.

First we parameterize: $\vec{r}(t) = t\vec{i} + t^2\vec{j}$ with $1 \leqslant t \leqslant 3$. Next, we have $\vec{x}'(t) = 1$ and $\vec{y}'(t) = 2t$, so

$$\int_C (xy + \ln(x))dy + xdx = \int_1^3 (t^3 + \ln(t))2tdt + \int_1^3 (t)1dt$$

$$= \int_1^3 (2t^4 + 2t\ln(t))dt + \int_1^3 tdt$$

(*ii*) Evaluate $\int_C (y/x)ds$ where C is along the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.

We have $\vec{r}(t) = t\vec{i} + t^2\vec{j}$, so $\vec{r}'(t) = \vec{i} + 2t\vec{j}$ giving $||\vec{r}'(t)|| = \sqrt{1+4t^2}$. Then we have:

$$\int_C (y/x)ds = \int_0^2 \frac{t^2}{t}\sqrt{1+4t^2}dt = \int_0^2 t\sqrt{1+4t^2}dt$$

$$= \left.\frac{(1+4t^2)^{\frac{3}{2}}}{12}\right|_0^2 = \frac{\sqrt{17}-1}{12}$$

(iii) Evaluate $\int_C (y/x)ds$ where C is along the line $y = 2x$ from $(0,0)$ to $(2,4)$.

We have $\vec{r}(t) = t\vec{i} + 2t\vec{j}$, so $\vec{r}'(t) = \vec{i} + 2\vec{j}$ giving $||\vec{r}'(t)|| = \sqrt{5}$. Then we have:

$$\int_C (y/x)ds = \int_0^2 \frac{t^2}{t}\sqrt{5}dt = \int_0^2 t\sqrt{5}dt$$
$$= \left.\frac{\sqrt{5}t^2}{2}\right|_0^2 = 2\sqrt{5}$$

Observations:

(i) In the last example we integrated $f(x,y) = y/x$ between the same two points along different paths and resulted in a different answer. This means that line integrals sometimes depend upon the path we take (we shall discuss this in detail in the next section).

(ii) Given a curve C, there are two directions we can travel - from left to right or right to left. In order to avoid this problem, instead of just giving a curve C, we also specify an **orientation**, or direction. Specifically, we say a curve C is oriented if a direction of travel has been specified. Note that if C is an oriented curve and $-C$ is the same curve oriented in the opposite direction, by the definition of line integral, we have

$$\int_{-C} f(x,y)ds = -\int_C f(x,y)ds$$
$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$$

and

$$\int_{-C} f(x,y)dy = -\int_C f(x,y)dy$$

2. Line Integrals of Scalar Functions over Space Curves

If $f(x,y,z)$ is a function of three variables which is continuous on the oriented space curve C, we can define line integrals with respect to distance, x, y and z and calculate them in exactly the same way. Specifically, we have the following:

Result 2.1. If $f(x,y,z)$ is a continuous function on a space curve C parameterized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ with $a \leqslant t \leqslant b$, then we have the following:

(i)

$$\int_C f(x,y,z)ds = \int_a^b (f(x(t),y(t),z(t)))||\vec{r}'(t)||dt$$

(*ii*)

$$\int_C f(x,y,z)dx = \int_a^b (f(x(t),y(t),z(t)))\vec{x}'(t)dt$$

(*iii*)

$$\int_C f(x,y,z)dy = \int_a^b (f(x(t),y(t),z(t)))\vec{y}'(t)dt$$

(*iv*)

$$\int_C f(x,y,z)dz = \int_a^b (f(x(t),y(t),z(t)))\vec{z}'(t)dt$$

(*v*)

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

$$= \int_a^b (f(x(t),y(t),z(t)))\vec{x}'(t)dt + \int_a^b (f(x(t),y(t),z(t)))\vec{y}'(t)dt + \int_a^b (f(x(t),y(t),z(t)))\vec{z}'(t)dt$$

The geometric interpretations are more difficult to visualize than the 2-d case, but the calculations are exactly the same. We illustrate with some examples.

Example 2.2. (*i*) Evaluate $\int_C xy^3 ds$ where $x = 4\sin t$, $y = 4\cos t$ and $z = 3t$ with $0 \leqslant t \leqslant \pi/2$.

Here we have $||\vec{r}'(t)|| = \sqrt{16(\cos t)^2 + 16(\sin t)^2 + 9} = 5$, so

$$\int_C xy^3 ds = 5\int_0^{\frac{\pi}{2}} 256 \sin t \cos^3 t dt = -320\cos^4 t\Big|_0^{\frac{\pi}{2}} = 320$$

(*ii*) Evaluate $\int_C x^2 y\sqrt{z}dz$ over the parameterized curve C with $\vec{r}(t) = t^3\vec{i} + t\vec{j} + t^2\vec{k}$, $0 \leqslant t \leqslant 1$.

Here we have $z'(t) = 2t$, so

$$\int_C x^2 y\sqrt{z}dz = \int_0^1 t^6 * t * t * 2tdt = \int_0^1 2t^9 dt = \frac{t^{10}}{5}\Big|_0^1 = \frac{1}{5}$$

3. Line Integrals of Vector Fields over Curves

Just as we defined a line integral of a scalar valued function, we can also define a line integral of a vector valued function, or a vector field. Since the output of a vector valued function is vectors as opposed to scalars, we need to modify our definition. We define it as follows:

(*i*) Suppose $\vec{F}$ is a vector field (either in space or in the plane) and C is a curve parameterized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ with $a \leqslant t \leqslant b$ (we omit the $z(t)$ in 2-space).

(*ii*) Break up the interval $[a,b]$ into n equal sized pieces of lengths $\Delta s = (b-a)/n$.

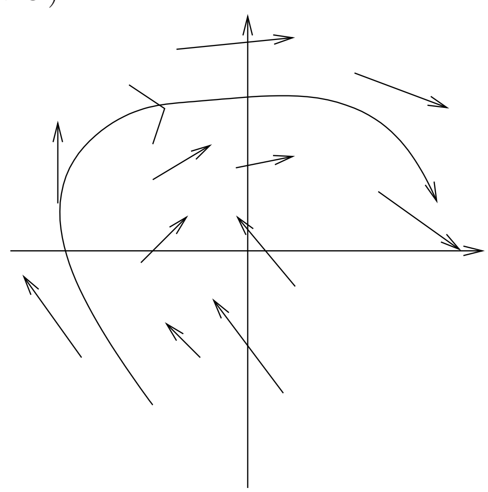
(*iii*) Fix a point $(\bar{x}_i, \bar{y}_i)$ on each of the segments of the curve.

(*iv*) We can form the Riemann sum

$$\sum_{i=1}^{n} \vec{F}(\bar{x}_i, \bar{y}_i) \cdot \vec{T}(\bar{x}_i, \bar{y}_i)\Delta s$$

where $\vec{T}$ denotes the unit normal tangent vector to C at a given point. Observe that this is a sum of scalars.

(*v*) Since we are taking a dot product, it will be positive if they point in the same general direction and negative else, so this Riemann sum measures how much the direction in which the curve is traveling agrees with the vector field $\vec{F}$ (in the illustration below, the Riemann sum would be large and positive because all the vectors are pointing in the same direction as the curve C).



Thus we define the line integral of a vector field over a curve as follows:

Definition 3.1. If $\vec{F}$ is a continuous vector field on a smooth curve C given by the vector equation $\vec{r}(t)$ with $a \leqslant t \leqslant b$, then we define the line integral of $\vec{F}$ along C as

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{n\to\infty} \vec{F}(\bar{x}_i, \bar{y}_i) \cdot \vec{T}(\bar{x}_i, \bar{y}_i)\Delta s.$$

It can be calculated using the following formula:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

where $\vec{F}(\vec{r}(t))$ simply denotes $\vec{F}(x(t), y(t), z(t))$ (the components of $\vec{r}(t)$).

Thus to calculate a line integral of $\vec{F}$ over C we do the following:

(*i*) Differentiate the parameterization $\vec{r}(t)$.

(*ii*) Compose the parameterization with $\vec{F}$ i.e $\vec{F}(\vec{r}(t))$.

(*iii*) Take the dot product $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$.

(*iv*) Integrate the resulting scalar function in t with respect to t between a and b.

Note that line integrals of vector fields are closely related to line integrals of functions. Specifically, we have the following result.

Result 3.2. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and C is parameterized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz$$

Proof. This is easy to prove. We have

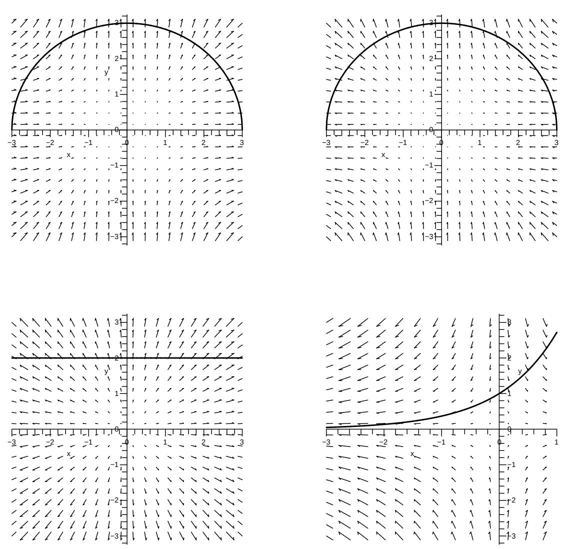
$$\int_C \vec{F} \cdot d\vec{r} = \int_C (P(\vec{r}(t))\vec{i} + Q(\vec{r}(t))\vec{j} + R(\vec{r}(t))\vec{k}) \cdot (x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k})dt$$

$$= \int_C (P(\vec{r}(t))x'(t) + Q(\vec{r}(t))y'(t) + R(\vec{r}(t))z'(t)dt = \int_C Pdx + Qdy + Rdz.$$

□

We finish with a couple of examples.

Example 3.3. (*i*) Determine whether the following line integrals are positive, negative or zero (all curves are oriented from left to right).



The first looks positive, the second and last negative, and the third zero (estimating which curves agree with the vector field).

(*ii*) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \sin(x)\vec{i} + \cos(y)\vec{j} + xz\vec{k}$ where $\vec{r}(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k}$ with $0 \leqslant t \leqslant 1$.

We have

$$\vec{r}'(t) = 3t^2\vec{i} - 2t\vec{j} + \vec{k}$$

and

$$\vec{F}(\vec{r}(t)) = \sin(t^3)\vec{i} + \cos(-t^2)\vec{j} + t^4\vec{k} = \sin(t^3)\vec{i} + \cos(t^2)\vec{j} + t^4\vec{k}.$$

Next we calculate the dot product:

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 3t^2 \sin(t^3) - 2t\cos(t^2) + t^4.$$

Thus we need to calculate

$$\int_0^1 3t^2 \sin(t^3) - 2t\cos(t^2) + t^4 dt = -\cos(t^3) - \sin(t^2) + \frac{t^5}{5}\Big|_0^1$$

$$= -\cos(1) - \sin(1) + \frac{1}{5} - (-1) = -\cos(1) - \sin(1) + \frac{6}{5}$$