Section 17.3
The Fundamental Theorem for Line Integrals

"FTC Number 2"

Recall that in single variable calculus, in order to evaluate a definite integral, we use the fundamental theorem of calculus. Specifically, we have

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Provided $F'(x)$ is a continuous function, an antiderivative always exists, so FTC can **always** be used to evaluate single variable integrals of continuous functions. In this section, we consider a generalization of FTC to line integrals. The difference in this case however is that we shall not always be able to apply it.

1. The Fundamental Theorem

In multivariable calculus, it doesn't make sense to talk about "the derivative" since a function of more than one variable can be differentiated with respect to different variables. In our earlier discussions on differentiation, we concluded that though we could not define "the derivative", we could define a generalization of the derivative - specifically, we said that the gradient vector of a function f was the closest generalization to "the derivative" we could get. With this in mind, we have the following generalization of FTC in single variable.

Result 1.1. Let C be a smooth curve parameterized by $\vec{r}(t)$ with $a \leqslant t \leqslant b$ and suppose f is a differentiable function of two or three variables whose gradient vector is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

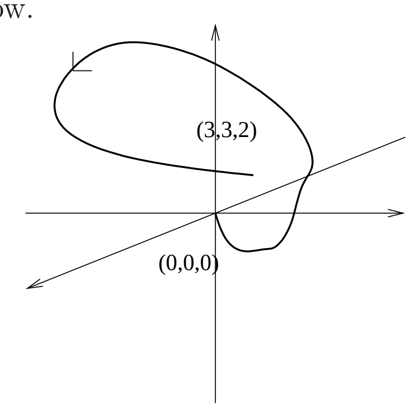
This results tells us that if $\vec{F}$ is a conservative vector field (meaning it is the gradient vector field of some function $f(x,y)$), then rather than evaluating the integral by hand, we can simply find the potential function and take its difference at the end points. We illustrate.

Example 1.2. (*i*) Suppose $\vec{F} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{r} = 3t^2\vec{i} + 3t^2\vec{j} + 2t\vec{k}$ with $0 \leqslant t \leqslant 1$.

Observe that $\vec{F}$ is a gradient vector field with potential function $f(x,y,z) = x^2 + y^2 + z^2$, so we can use FTC to evaluate the integral. Since the end points are $(0,0,0)$ and $(3,3,2)$, we have

$$\int_C \vec{F} \cdot d\vec{r} = f(3,3,2) - f(0,0,0) = 9 + 9 + 4 = 22$$

(*ii*) Suppose $\vec{F} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{r}$ is drawn below.



Since this is a conservative vector field, we can use FTC. Since the end points are $(0,0,0)$ and $(3,3,2)$, we have

$$\int_C \vec{F} \cdot d\vec{r} = f(3,3,2) - f(0,0,0) = 9 + 9 + 4 = 22$$

In the last two examples, observe that the answer was the same. This is simply because the vector field $\vec{F}$ is conservative, so the integral can be evaluated using FTC. This observation motivates the following definition.

Definition 1.3. A vector field $\vec{F}$ is called **path independent** if

$$\int_{C_1} \vec{F} \cdot D\vec{r} = \int_{C_1} \vec{F} \cdot D\vec{r}$$

for any two paths C_1 and C_2 between any two points.

Our observations prove the following.

Result 1.4. A conservative vector field $\vec{F}$ is path independent.

Since we can apply FTC when a vector field is path independent it makes integrals much easier. Also, we can apply FTC when a vector field is conservative, so it leads to two natural questions: Are path independent fields conservative and, how can we tell if a vector field is path independent?

2. Path Independence

In the last section, we showed that if a vector field is conservative, then it is path independent. It is natural to ask the converse - that is, if a vector field is path independent, is it a conservative vector field. In order to answer this questions, we need to explore the problem in a but more detail. We start with some definitions.

Definition 2.1. A curve C is called closed if its terminal point coincides with its endpoint.

The notion of path independence can be reformulated into a much nicer definition using the concept of closed curves.

Result 2.2. A vector field $\vec{F}$ is path independent if and only if

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

for **any** closed curve C.

Proof. The proof of this is easy. First, suppose $\vec{F}$ is path independent and let C be some closed curve. We need to show $\int_C \vec{F} \cdot d\vec{r} = 0$. However, we can split C up into two curves, C_1 and C_2. Observe that since $\vec{F}$ is path independent, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} d\vec{F} \cdot d\vec{r}$$

where $-C_2$ denotes the path along C_2 oriented in the opposite direction. Then we have

$$0 = \int_{C_1} d\vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}$$

Now suppose $\int_C \vec{F} \cdot d\vec{r} = 0$ for any curve C. We need to show $\vec{F}$ is path independent. If C_1 and C_2 are paths with the same intial and terminal points, we can form a closed path $C = C_1 - C_2$, so

$$0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r}$$

where $-C_2$ denotes the path along C_2 oriented in the opposite direction. Then we have

$$0 = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

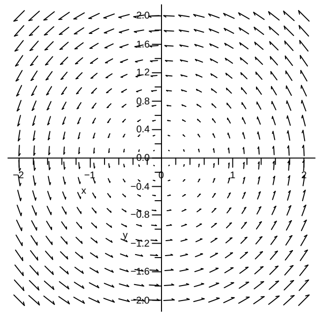
and so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

□

This means to show something is not path independent, it suffices to show that there exists some closed curve for which the line integral is non-zero (however, this does not give us a way to conclude that something is path independent!). We illustrate.

Example 2.3. Explain why the following field is not conservative.



Let C denote the unit circle centered at the origin oriented counter-clockwise. Then

$$\int_C \vec{F} d\vec{r} > 0$$

since all the arrows point in the same direction as $\vec{F}$. Hence the integral will be positive, so $\vec{F}$ cannot be path indpendent (since the integral over a closed curve would have to be zero).

We are almost ready to answer the question relating path independence and conservative vector fields. First, we need two more definitions.

Definition 2.4. We call a set D open if every point in the set can be surrounded by a small disc which is entirely contained in D.

Definition 2.5. We call a set D connected if any two points in D can be connected by a path.

Result 2.6. Suppose that $\vec{F}$ is a vector field which is continuous on an open and connected region D. If $\vec{F}$ is path independent on D then $\vec{F}$ is conservative on D.

This means that for most path independent vector fields we consider, they will be conservative (that is, they will be the gradient vector of some function).

3. Showing a 2D Vector Field is Conservative

If we can show a vector field is conservative, it means we can use FTC to evaluate an integral and answer problems much quicker. However we do not yet have a way to determine when a vector field is conservative, so this will be our next task. For the rest of the section, we shall assume that $\vec{F}$ is a 2 dimensional vector field - we shall consider a different result for 3 dimensional vector fields later. Before we determine a test, we observe the following consequence of Clairaut's Theorem for conservative vector fields.

Result 3.1. Suppose $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field where P and Q have continuous first order partial derivatives on a domain D. Then on D, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

With this result in mind, we can determine a test to see if a vector field is conservative. For this we need a definition.

Definition 3.2. A region D is called simply connected if given any closed curve in C, it only contains points which are in D. Equivalently, a region is not simply connected if there is some closed curve which can be drawn in D with the property that there are points **not in** D inside the curve.

For 2-space, simply connected basically means there are no missing regions from the space. The following will allow us to determine whether a vector field is conservative.

Result 3.3. Let $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field on an open, simply connected region D. Suppose P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. Then $\vec{F}$ is conservative.

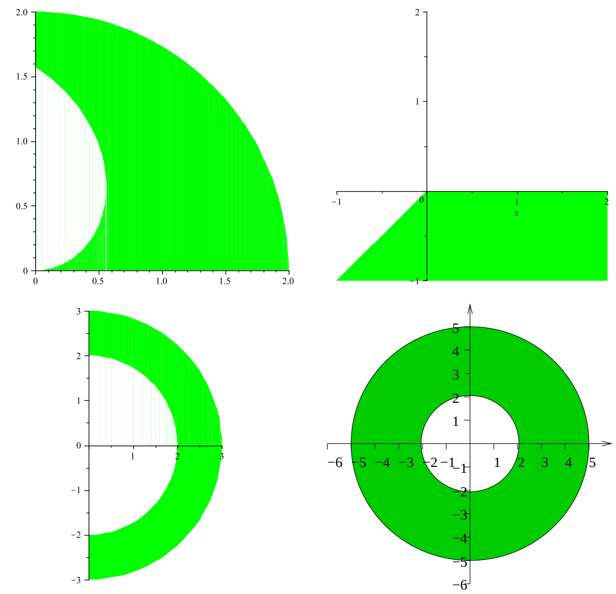
Remark 3.4. We sometimes call the function

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

the 2 dimensional curl of $\vec{F}$. Note that if the 2 dimensional curl of a vector field is 0 on a simply connected domain and the partial derivatives are continuous, then the vector field is conservative.

We finish with some examples illustrating the results we have proved.

Example 3.5. Which of the following regions are simply connected? All three of these regions are simply connected - there is no way to draw a closed curve in any of them which contains points which are not in the region themselves. The last region however is not simply connected - we can sketch a circle centered at the origin within this region and it will contain points which are no in the region itself.



Example 3.6. Determine which of the following are conservative and if they are, find a potential function.

(*i*) $\vec{F} = xe^y\vec{i} + ye^x\vec{j}$

The 2 dimensional curl is:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = ye^x - xe^y \neq 0,$$

so it is not conservative.

(*ii*) $\vec{F} = e^y\vec{i} + xe^y\vec{j}$

The 2 dimensional curl is:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^y - e^y = 0,$$

so it will be conservative on any simply connected region. Since it is defined for all values of x and y, it will be conservative on the whole plane $\mathbb{R}^2$. A potential function is $f(x, y) = xe^y$.

Example 3.7. Suppose that $\vec{F}(x, y, z) = y^2\cos(z)\vec{i} + 2xy\cos(z)\vec{j} - xy^2\sin(z)\vec{k}$. If C is parameterized by $\vec{r} = t^2\vec{i} + (\sin(t))\vec{j} + (t)\vec{j}$ with $0 \leqslant t \leqslant \pi$, determine $\int_C \vec{F} \cdot d\vec{r}$.

Before we do anything, we check whether it is conservative. However, observe that $f(x, y, z) = xy^2\cos(z)$ is a potential function for $\vec{F}$, so it is conservative. Thus we can apply FTC. Since the initial point of C is $(0, 0, 0)$ and the terminal point is $(\pi^2, 0, \pi)$, we have

$$\int_C \vec{F} \cdot d\vec{r} = 0 - 0 = 0$$

Example 3.8. Evaluate

$$\int_C \left(-\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}\right) d\vec{r}$$

where C is the unit circle centered at the origin oriented counter clockwise.

Observe that

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

so it seems that we should be able to apply FTC to conclude that the integral is 0. However, parameterizing, we have $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$ with $0 \leqslant t \leqslant 2\pi$, so $\vec{r}'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}$. Calculating, we have

$$\int_C \left(-\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}\right) d\vec{r}$$

$$= \int_0^{2\pi} \left(-\frac{\sin(t)}{\cos^2(t)+\sin^2(t)}\vec{i} + \frac{\cos(t)}{\cos^2(t)+\sin^2(t)}\vec{j}\right) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j}) dt$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = \int_0^{2\pi} 1 dt = 2\pi$$

which is not 0. This seems to be a problem and seems to contradict both FTC and the curl test. If we are a little more careful in our analysis however, we can see why in fact it does neither of these things.

First, the curl test does not apply since the domain of $\vec{F}$ is not a simply connected region - in particular, it excludes the origin. This means that on any region which contains the origin, $\vec{F}$ is not a necessarily a gradient vector on that region, and hence may not be path independent.

Next, observe that

$$\vec{F} = \nabla f$$

where

$$f(x,y) = \arctan\left(\frac{y}{x}\right),$$

so it seems that $\vec{F}$ is indeed a gradient vector field. However, to apply FTC, we need $f(x,y)$ to be continuous and differentiable on all of C. In this case, it is not since it is not defined at $y = 0$, so FTC does not apply. Hence our initial calculations are true since neither FTC nor the curl test apply to show it is a gradient vector field on the necessary region.