Section 17.4
Green's Theorem

"Calculating Line Integrals using Double Integrals"

In the previous section, we saw an easy way to determine line integrals in the special case when a vector field $\vec{F}$ is conservative. This still leaves the problem of finding a line integral over a vector field which is not conservative. In this section we shall consider a way to determine line integrals of vector fields which are not necessarily conservative over closed curves.

1. Greens Theorem

Green's Theorem gives us a way to transform a line integral into a double integral. To state Green's Theorem, we need the following definition.

Definition 1.1. We say a closed curve C has positive orientation if it is traversed counterclockwise. Otherwise we say it has a negative orientation.

The following result, called Green's Theorem, allows us to convert a line integral into a double integral (under certain special conditions).

Result 1.2. (Green's Theorem) Let C be a positively oriented piecewise smooth simple **closed** curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region which contains D and $\vec{F} = P\vec{i} + Q\vec{j}$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Note that we are integrating the curl of the vector field $\vec{F}$, so we sometimes write Green's Theorem as:

$$\int_C \vec{F} \cdot d\vec{r} = \int_D \mathrm{curl}(\vec{F}) dA.$$

Notation: Sometimes we use the notation ∂D to denote C (meaning the boundary of D).

Notation: Sometimes we use the notation $\oint$ to denote a line integral over a **closed** curve.

We illustrate Green's Theorem with some examples.

Example 1.3. Evaluate the line integral $\oint y dx - x dy$ where C is the unit circle centered at the origin oriented counterclockwise both directly and using Green's Theorem.

(*i*) To evaluate it directly, we first note that C is parameterized by $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$ with $0 \leqslant t \leqslant 2\pi$. Then we have $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$, so we get

$$\oint ydx - xdy = \int_0^{2\pi} (\sin(t))(-\sin(t))dt - \int_0^{2\pi} (\cos(t))(\cos(t))dt$$
$$= -\int_0^{2\pi} (\cos^2(t) + \sin^2(t))dt = -2\pi$$

(*ii*) To evaluate using Green's Theorem, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2$. Since the curve is oriented counterclockwise, we can apply Green's Theorem:

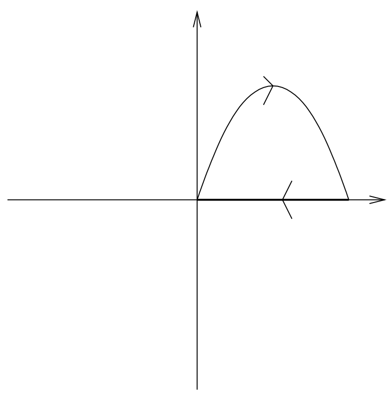
$$\oint_C ydx - xdy = \int\int_D -2dA = -2\int_0^{2\pi}\int_0^1 rdrd\vartheta = -4\pi\frac{r^2}{2}\bigg|_0^1 = -2\pi$$

Example 1.4. Use Green's Theorem to evaluate

$$\int_C (\sqrt{x} + y^3)\vec{i} + (x^2 + \sqrt{y})\vec{j} \cdot \vec{r}$$

where C is the curve $y = \sin(x)$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$.

First observe that this curve is oriented in the wrong direction (see illustration).



However, as we observed before, we have $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$, so we need to find the line integral and then simply negate the answer.

Using Greens Theorem, we have

$$\oint_{-C} \vec{F} \cdot d\vec{r} = \int_0^{\pi}\int_0^{\sin(x)} (2x - 3y^2)dydx$$
$$= \int_0^{\pi} (2xy - y^3)\bigg|_0^{\sin(x)} dx = \int_0^{\pi} 2x\sin(x) - (\sin(x))^3 dx$$
$$= \int_0^{\pi} 2x\sin(x) - (\sin(x))^3 dx = 2\pi - \frac{4}{3}$$

(integration by parts and trig integrals).

Example 1.5. Let

$$\vec{F} = \frac{y}{x^2+y^2}\vec{i} - \frac{x}{x^2+y^2}\vec{j}$$

and observe that $\nabla f = \vec{F}$. If $\vec{r} = \cos(t)\vec{i} + \sin(t)\vec{j}$ with $0 \leqslant t \leqslant 2\pi$, calculate

$$\int_C \vec{F} \cdot d\vec{r}$$

directly and then explain why you cannot calculate it using either FTC or Green's Theorem.

(i) Directly we have

$$\int_C \nabla f \cdot d\vec{r} = \int_0^{2\pi} \left(\frac{y}{x^2+y^2}\vec{i} - \frac{x}{x^2+y^2}\vec{j} \right) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})dt$$

$$= \int_0^{2\pi} \Big(\sin(t)\vec{i} - \cos(t) \Big) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})dt = \int_0^{2\pi} -\sin^2(t) - \cos^2(t)dt = -2\pi$$

(ii) If we were to try to use Greens Theorem, we would have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0,$$

so Green's Theorem would say

$$\int_C \vec{F} \cdot d\vec{r} = \int_D 0dA = 0$$

(iii) If we were to try to use the Fundamental Theorem, since $\vec{F} = \nabla(\arctan(x))$, the fundamental theorem would say

$$\int_C \vec{F} \cdot d\vec{r} = \arctan(0) - \arctan(0) = 0$$

Note that the second two answers do not match up with the direct calculation, and so something must have gone wrong? Notice that $\arctan(x/y)$ is not defined at the points $(1,0)$ and $(-1,0)$, and $\vec{F}$ is not defined at $(0,0)$, In particular, in order to apply Green's Theorem, $\vec{F}$ must be defined inside and on C, and in order to apply FTC, $f(x,y)$ must defined on C - therefore, in this case we cannot apply Green's theorem or FTC, and the only way to calculate this integral is directly.

We can also go in the other direction - Green's Theorem can be used to turn a double integral into a line integral which sometimes may be more useful. One particular application is in finding the area of a region. Specifically, we have the following useful result:

Result 1.6. If D is a closed simple region, then the area of D is given by

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint xdy - ydx.$$

We illustrate with an example.

Example 1.7. Find the area under one arch of the cycloid $x = t - \sin(t)$, $y = 1 - \cos(t)$.

One arc of the centroid (oriented clockwise) occurs for $0 \leqslant t \leqslant 2\pi$, so we need to calculate $\int\int_D 1dA$ where D is the region bounded by this parametric equation and below by $y = 0$. Using Greens Theorem, it suffices to calculate

$$\oint_C xdy$$

where C is the curve bounded by the cycloid and $y = 0$. Let C_1 denote the part of the cycloid and C_2 the horizontal component along the axis. Since C_1 is oriented clockwise, we have

$$\int_C xdy = -\oint_{C_1} xdy + \oint_{C_2} xdy$$

A parameterization of C_2 is $\vec{r}(t) = (2\pi - t)\vec{i}$ with $0 \leqslant t \leqslant 2\pi$, so we have

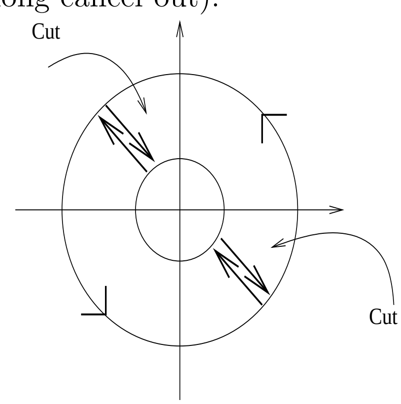
$$\oint_{C_2} xdy = \int_0^{2\pi} (2\pi - t)0dt = 0.$$

Then for C_1, we have

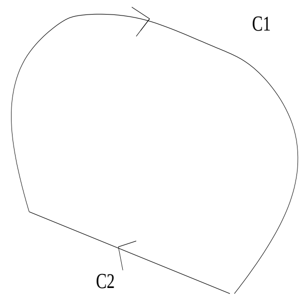
$$\oint_{C_1} xdy = \int_0^{2\pi} (t - \sin(t))\sin(t)dt = -3\pi.$$

It follows that the area will be 3π.

Green's Theorem can actually be modified to integrate over regions which contain holes by simply breaking a region up into smaller pieces so each of them are simply connected (observe that the integrals along the curves we cut along cancel out).



Green's Theorem can also be used indirectly to calculate line integrals of a vector field $\vec{F}$ over curves which are not necessarily closed. Specifically, if we have a line integral over the complicated curve C_1 illustrated below, then we can convert it into a sum of a double integral and a line integral over a much easier curve C_2 illustrated below using Green's Theorem:



Specifically, we have

$$\int_{C_1} \vec{F} d\vec{r} + \int_{C_2} \vec{F} d\vec{r} = \int\int_D \mathrm{curl}(\vec{F}) dA$$

or equivalently

$$\int_{C_1} \vec{F} d\vec{r} = \int\int_D \mathrm{curl}(\vec{F}) dA - \int_{C_2} \vec{F} d\vec{r}.$$

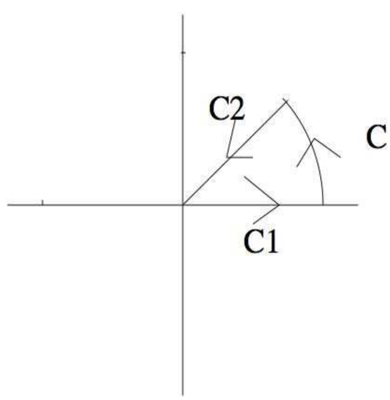
We illustrate with an example.

Example 1.8. Find the line integral

$$\int_C ((x-y)\vec{\imath} + x\vec{\jmath}) \cdot d\vec{r}$$

where C is the segment of the circle $x^2 + y^2 = 9$ with $0 \leqslant \vartheta \leqslant \pi/2$ oriented counterclockwise.

Notice that if we parametrize this portion of the circle and evaluate this integral, we get a very messy trig integral. Therefore, we shall try to use Green's Theorem indirectly. Let C_1 be the line segment from the origin to the edge of the circle along the x-axis, and let C_2 be the line from the end of the segment back to the origin (see illustration below).



Then using Green's Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} = \int\int_D \mathrm{curl}(\vec{F}) dA,$$

and we can calculate each of the integrals individually. Specifically,

$$\int\int_D \text{curl}(\vec{F})dA = \int\int_D 1-(-1)dA = \int\int_D 2dA = 2\times\frac{9\pi}{8} = \frac{9\pi}{4}$$

(since we are integrating a constant, it will simply be the area of the region times the constant). For C_1, a parametrization will be $\vec{r}(t) = t\vec{i}$ for $0 \leqslant t \leqslant 3$. Thus,

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^3 (t\vec{i}+t\vec{j})\cdot(\vec{i})dt = \int_0^3 2tdt = \frac{t^2}{2}\bigg|_0^3 = \frac{9}{2}.$$

For C_2, a parameterization will be $\vec{r}(t) = (3-t)\vec{i}+(3-t)\vec{j}$ for $0 \leqslant t \leqslant 3$, so

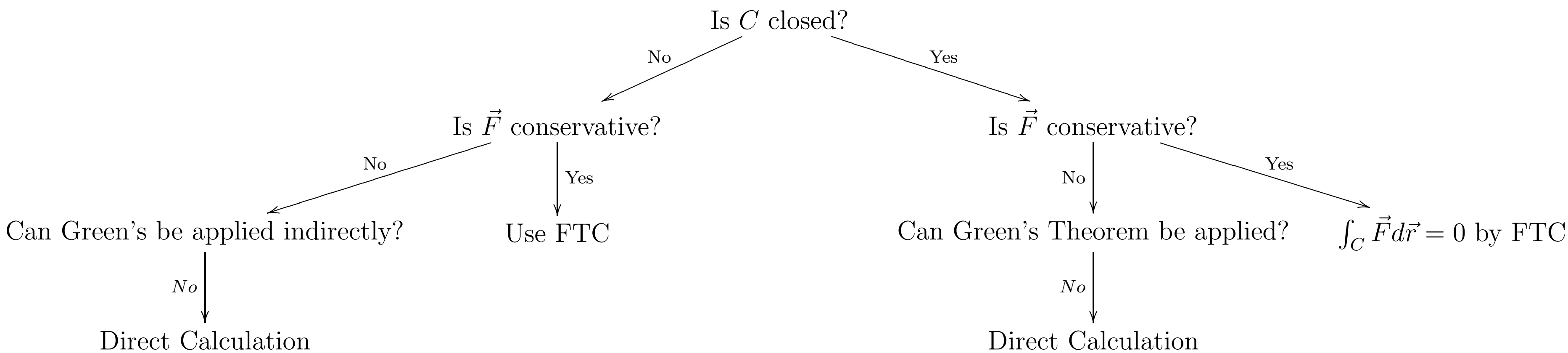
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^3 (0\vec{i}+(3-t)\vec{j})\cdot(-\vec{i}-\vec{j})dt = \int_0^3 (t-3)dt = \frac{t^2}{2}-3t\bigg|_0^3 = \frac{9}{2}-9 = -\frac{9}{2}.$$

Therefore,

$$\int_C \vec{F}\cdot d\vec{r} = \frac{9\pi}{4} - \frac{9}{2} + \frac{9}{2} = \frac{9\pi}{4}.$$

2. Calculating Line Integrals: A Summary

We have seen a lot of different ways to calculate line integrals, so it would be a good idea to pause and try to reflect on the different methods we have considered, and which method to use under which circumstances. Ultimately, this choice comes down to two simple questions: Is C closed? and Is $\vec{F}$ conservative? Ideally, we never want to calculate a line integral directly because it requires a lot of work, so answering these questions is crucial in making the calculations as easy as possible. As a general case, the decision tree on the following page is a good indicator of which method should be used under which circumstances.


Is C closed?
No
Yes
Is $\vec{F}$ conservative?
Is $\vec{F}$ conservative?
No
Yes
No
Yes
Can Green's be applied indirectly?
Use FTC
Can Green's Theorem be applied?
$\int_C \vec{F} d\vec{r} = 0$ by FTC
No
No
Direct Calculation
Direct Calculation