Section 17.5
Curl and Divergence

"New Tools for Line Integrals"

In the last section, we used the 2-d curl to transform a line integral over a closed curve into a double integral over the region inside the curve. We also used the 2-d curl to determine whether a vector field was conservative. In this section we consider two new functions (one a scalar function and the other a vector function) which we shall be able to use to transform complicated vector integrals into much more straightforward integrals over regions and determine whether a vector field is conservative.

1. Curl

We start with a definition.

Definition 1.1. Suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a differentiable vector field. Then we define the curl of $\vec{F}$ as the vector function

$$\text{curl}(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

The formula for the curl is very long, so it would be useful to have a was to remember it. This can be done through a cross product. Specifically, if we define

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k},$$

then we have

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}.$$

We consider an example.

Example 1.2. Calculate the curl of $\vec{F} = xyz\vec{i} - x^2y\vec{k}$.

Using the formula, we have

$$\text{curl}(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$
$$= (-x^2 - 0)\vec{i} + (xy + 2xy)\vec{j} + (-xz)\vec{k} = -x^2\vec{i} + 3xy\vec{j} - xz\vec{k}$$

The following theorem is a generalization of the 2-d case and is a consequence of Clairuts Theorem.

Result 1.3. If f is a function of three variables that has continuous second order partial derivatives, then

$$\text{curl}(\nabla f) = \vec{0}$$

More importantly, this gives us a way of determining whether or not a 3-d vector field is conservative.

Result 1.4. If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is a conservative vector field.

We consider an example.

Example 1.5. Show that the vector field $\vec{F} = e^x \sin(y)\vec{i} + e^x \cos(y)\vec{j} + z\vec{k}$. is conservative and determine a potential function.

Clearly the partial derivatives are continuous, so using the formula, we have

$$\text{curl}(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (0-0)\vec{i} + (0-0)\vec{j} + (e^x \cos(y) - e^x \cos(y))\vec{k} = \vec{0},$$

so the vector field is conservative. A potential function for this vector field is

$$f(x, y, z) = e^x \sin(y) + \frac{z^2}{2}.$$

2. Divergence

Next we introduce a new idea which will be important in future sections.

Definition 2.1. Suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a differentiable vector field. Then we define the divergence of $\vec{F}$ as the scalar function

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

With ∇ as defined before, the divergence can be remembered via the formula

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F}.$$

We consider an example.

Example 2.2. Calculate the divergence of $\vec{F} = xyz\vec{i} - x^2y\vec{k}$.

Applying the formula, we have

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = yz$$

The following result again uses Clairats Theorem and gives us a way to test whether a vector field $\vec{F}$ is the curl is some other vector field $\vec{G}$ - it will be important in later sections.

Result 2.3. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and P, Q and R have continuous second-order partial derivatives, then

$$\text{div}(\text{curl}\vec{F}) = 0.$$

This tells us that if

$$\text{div}(\vec{F}) \neq 0$$

then $\vec{F}$ cannot be the curl of some vector field $\vec{G}$ (it is not a curl field).

Example 2.4. Show that $\vec{F} = xyz\vec{i} - x^2y\vec{k}$ is not a curl field.

As before, we have

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = yz \neq 0,$$

so it cannot be a curl field.

3. The Vector Form of Green's Theorem

Recall that Green's Theorem states that

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

We referred to the functions $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ as the 2-d curl, and the main reason for doing so is because Green's Theorem can be restated in terms of the 3-d curl. Specifically, we have the following:

Result 3.1.

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_D (\text{curl}\vec{F}) \cdot \vec{k} dA.$$

This formula gives us a nice way to remember Green's Theorem.