Section 17.6
Parametric Surfaces

"Parametric Equations Defining Surfaces"

In Chapter 14, we discussed how curves could be represented in space through the use of parametric equations in one variable. In this section, we consider the same idea for surfaces.

1. Parametric Surfaces

Suppose that $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ is a vector valued function defined on a region D in 3-space. For each value of (u,v), we associate the point $(x(u,v), y(u,v), z(u,v))$ to the vector $\vec{r}(u,v)$. Then as (u,v) vary over D, the vector function $\vec{r}(t)$ traces out a surface S called the parametric surface with parametric equations $\vec{r}(u,v)$. We illustrate.

Example 1.1. Identify the surface with parametric equations

$$\vec{r}(u,v) = u\cos(v)\vec{i} + u\sin(v)\vec{j} + u^2\vec{k}.$$

Since $x = u\cos(v)$, $y = u\sin(v)$ and $z = u^2$, at any point on this surface we have $x^2 + y^2 = u^2 = z$. This is the equation for a parabolic bowl centered on the z-axis with vertex at the origin.

Example 1.2. Identify the surface with parametric equations

$$\vec{r}x, \vartheta) = u\vec{i} + u\cos(\vartheta)\vec{j} + u\sin(\vartheta)\vec{k}.$$

Since $x = x$, $y = x\cos(\vartheta)$ and $z = x\sin(\vartheta)$, at any point on this surface we have $y^2 + z^2 = x^2$. This is the equation for a cone centered on the x-axis with vertex at the origin.

Usually parametric surfaces are much more difficult to describe. We can usually get a good idea by looking at a small number of points though often a good drawing will require the use of a calculator or computer algebra system like Maple.
In addition to sketching parametric surfaces, it is also very important to be able to determine equations for parametric surfaces. There are some surfaces for which this is easy and others which are much more difficult. We illustrate with some examples.

Example 1.3. Suppose S is a surface which is the graph of some function $z = f(x,y)$ over some region D in the plane. Then a parameterization for S is $\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$.

Example 1.4. Find parametric equations for the following surfaces

(*i*) The lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$.

Since z is a function of x and y, we have $z = \sqrt{1 - 2x^2 - 4y^2}$, so a parameterization

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + \sqrt{1 - 2x^2 - 4y^2}\vec{k}$$

where $2x^2 + 4y^2 \leqslant 1$.

(*ii*) The part of the sphere $x^2 + y^2 + z^2 = 16$ which lies between the planes $z = 2$ and $z = -2$.

Here we can use spherical coordinates to help us. Since we are considering a sphere of radius 4, we have $\varrho = 4$ and $x = 4\sin(\varphi)\cos(\vartheta)$, $y = 4\sin(\varphi)\sin(\vartheta)$, $z = 4\cos(\varphi)$. Since $-2 \leqslant z \leqslant 2$, we have $-2 \leqslant 4\cos(\varphi) \leqslant 2$ or $\pi/3\varphi \leqslant 2\pi/3$, so a parameterization is

$$\vec{r}(\vartheta, \varphi) = 4\sin(\varphi)\cos(\vartheta)\vec{i} + 4\sin(\varphi)\sin(\vartheta)\vec{j} + 4\cos(\varphi)\vec{k}$$

with $0 \leqslant \vartheta \leqslant 2\pi$ and $\pi/2 \leqslant \varphi \leqslant 2\pi/3$.

(*iii*) The part of the plane $z = x + 3$ that lies inside the cyclinder $x^2 + y^2 = 1$.

Since z is a function of x, we have the parameterization

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + (x + 3)\vec{k}$$

with the restriction $x^2 + y^2 \leqslant 1$. Alternatively, we could use polar coordinates with the bounds giving

$$\vec{r}(r, \vartheta) = r\cos(\vartheta)\vec{i} + r\sin(\vartheta)\vec{j}_+(r\cos(\vartheta) + 3)\vec{k}$$

with $0 \leqslant \vartheta \leqslant 2\pi$ and $0 \leqslant r \leqslant 1$.

2. Parametric Surfaces and Tangent Planes

Suppose S is a parametric surface given by the parameterization $\vec{r}(u, v)$. Then it is a fairly straight forward problem to find its tangent plane at a point. Specifically, if $\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$ and $\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$, then we have the following result:

Result 2.1. The vector $\vec{r}_u \times \vec{r}_v$ is a normal vector to the surface S parameterized by $\vec{r}(u, v)$.

With a normal vector and a point, we can write down the equation for a tangent plane. We finish with an example.

Example 2.2. Find a tangent plane to the surface with parametric equations $x = u$, $y = u^2$, $z = u - v^2$ at the point $(1, 1, 1)$.
We have $\vec{r}_u = \vec{i} + 2u\vec{j} + \vec{k}$ and $\vec{r}_v = -2v\vec{k}$. At the point $(1, 1, 1)$, we have $\vec{r}_u = \vec{i} + 2\vec{j} + \vec{k}$ and $\vec{r}_v = -2\vec{k}$, so $\vec{r}_u \times \vec{r}_v = -4\vec{i} + 2\vec{j}$. Thus an equation for the plane at this point will be

$$-4(x - 1) + 2(y - 1) = 0.$$