Section 17.7
Surface Integrals

"Integrating Functions over Arbitrary Surfaces"

In the last few sections we were considering integrals of functions of two or three variables over arbitrary curves. In this section we consider the integral of a function of three variables over a surface. Note that usually we integrate functions of three variables over regions in 3-space.

1. Surface Integrals

The definition of a surface integral is analogous to many of the other definitions of integrals we have given - it will require partitioning the surface into small rectangles and then constructing a Riemann sum over those rectangles. We proceed as follows.

(i) Suppose $f(x, y, z)$ is some function of three variables and S is a piecewise smooth surface in 3-space on which f is continuous.
(ii) Divide S up into small patches of area ΔS.
(iii) In each patch, fix a point (x_i, y_j, z_k).
(iv) Construct the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_j, z_k)\Delta S.$$

(v) We define the **Surface Integral** of $f(x, y, z)$ over S to be the limit of the sum

$$\int\int_S f(x, y, z)dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_j, z_k)\Delta S$$

provided this limit exists.

Surfaces have been described in many different ways, and accordingly, we have many different ways to evaluate surface integrals depending upon description. We shall look at the most important ways.

1.1. **Graphs of Functions.** Suppose that the surface S is the graph of some function $z = g(x, y)$ over the region D in the xy-plane. Then the surface integral of $f(x, y, z)$ over S can be calculated as

$$\int\int_S f(x, y, z)dS = \int\int_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}dA.$$

We illustrate with some examples.

Example 1.1. Evaluate the surface integral

$$\int\int_S xydS$$

where S is the triangular region with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,2)$.

Observe that this is a triangular portion of a plane which is the graph of a function. To find the function, we need to find the equation of the plane: we have $\vec{u} = \vec{i} - 2\vec{j}$ and $\vec{v} = \vec{i} - 2\vec{k}$, so the normal vector to the plane will be $\vec{n} = 4\vec{i} + 2\vec{j} + 2\vec{k}$, so the equation will be

$$4(x-1) + 2y + 2z = 0$$

or

$$g(x,y) = 2 - 2x - y.$$

Looking at the region D in the plane, we are integrating over $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant -2x+2$. Applying the formula, we have

$$\begin{aligned}
\int\int_S f(x,y,z)dS &= \int\int_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}dA \\
&= \int_0^1 \int_0^{-2x+2} xy\sqrt{4+1+1}dA = \sqrt{6}\int_0^1\int_0^{-2x+2} xydydx \\
&= \frac{\sqrt{6}}{2}\int_0^1 \left[xy^2\right]_0^{-2x+2} dx = \frac{\sqrt{6}}{2}\int_0^1 (4x^3 - 8x^2 + 4x)dx \\
&= \frac{\sqrt{6}}{2}\left[x^4 - \frac{8x^3}{3} + 2x^2\right]_0^1 = \frac{\sqrt{6}}{6}
\end{aligned}$$

2. Oriented Surfaces

As with curves, in order to define surface integrals of vector fields, we need to introduce the notion of orientation of surfaces. Unlike curves however, we shall see that there are surfaces which are not orientable. The formal definition of an oriented surface is the following.

Definition 2.1. We say a surface S is orientable if it is possible to choose a **unit** normal vector $\vec{n}$ at every point so that $\vec{n}$ varies continuously over S. If S is orientable, we call such a choice of normal vectors an orientation of S, and once an orientation of S has been specified, it is called an oriented surface.

For the two different surfaces we have considered, there are standard normal vectors we can define.

Definition 2.2. If a surface S is the graph of some function $g(x,y)$, then we choose the unit normal vector

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + 1}}$$

at each point to define an orientation on S.

All the surfaces we consider will be orientable (else the definition of the integral becomes problematic). There are however classic examples of non orientable surfaces - the Mobiüs strip and the Klein bottle. Neither of these surfaces are orientable because they each only have one side (so if we try to continuously move the normal vector around the surface, when we return to our initial point, the normal vector will have switched to the ”other side” of the surface).

3. Surface Integrals of Vector Fields

We are now ready to define the surface integral of a vector field in 3-space over a surface S. As with all integrals we have so far defined, the definition returns to the idea of Riemann sums, though since the output of a vector field is a vector, the sum Riemann sum needs to be modified accordingly. Rather than stating all the steps as we have with other integrals, since the construction is very similar, we shall just directly define a surface integral.

Definition 3.1. If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S (\vec{F} \cdot \vec{n}) dS.$$

The integral is also called the flux of $\vec{F}$ across S.

In order to determine what a flux integral measures, we need to consider what the quantity $\vec{F} \cdot \vec{n}$ measures at a point on the surface. However, this is similar to the line integral case: for the line integral case, if $\vec{F} \cdot \vec{r} > 0$, then $\vec{r}$ points roughly in the same direction as $\vec{F}$ (so the arrows of $\vec{F}$ are pointing through the surface in the direction of $\vec{r}$); likewise for surface intergals, if $\vec{F} \cdot \vec{n} > 0$, then $\vec{n}$ points roughly in the same direction as $\vec{F}$ (so the arrows of $\vec{F}$ are pointing through the surface in the direction of $\vec{n}$). Therefore, loosely speaking, the flux of a vector field $\vec{F}$ over a surface S can be thought of as how much the vector field passes **through** the surface in the direction of the orientation.

As with surface integrals of functions, we can derive a special formula for flux integrals over a surface given as the graph of a function.

3.1. **Graphs of Functions.**

Suppose that the surface S is the graph of some function $z = g(x, y)$ with **upward orientation** over the region D in the xy-plane. Then the surface integral of $\vec{F}(x, y, z)$ over S can be calculated as

$$\int\int_S \vec{F}(x, y, z) \cdot d\vec{S}$$

$$= \int\int_D \left(-P(x, y, g(x, y)) \frac{\partial g}{\partial x} - Q(x, y, g(x, y)) \frac{\partial g}{\partial y} + R(x, y, g(x, y)) \right) dA$$

(for downward orientation, we multiply by -1). We illustrate with an example.

Example 3.2. Find the flux integral

$$\int\int_S (xy\vec{i} + yz\vec{j} + xz\vec{k}) \cdot d\vec{S}$$

where S is the part of the parabola $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ oriented upward.

Using the formula for flux through the graph of a surface $g(x, y) = 4 - x^2 - y^2$, we have

$$\int\int_S \vec{F}(x, y, z) \cdot d\vec{S} = \int\int_D \Big(- P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\Big) dA$$

$$= \int_0^1 \int_0^1 -(xy)(-2x) - y(4 - x^2 - y^2)(-2y) + x(4 - x^2 - y^2) dx dy$$

$$= \int_0^1 \int_0^1 2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2 dx dy = \frac{713}{180}$$

All these examples show that surfaces integrals are VERY COMPLICATED and take a long time to calculate. There must be an easier way!!!!!