Section 17.8
Stokes' Theorem

"Making Flux Integrals Easy"

Our first approach to line integrals was the brute force method - parameterize the curve, take the dot product and then integrate. Though this method is always guaranteed to work, we saw very quickly that it can take a very long time to do some fairly simple integrals. This led us to two other ways to calculate line integrals - Green's Theorem and the Fundamental Theorem of Calculus for line integrals, both methods which were much easier than brute force. In the next two sections we shall consider two different methods to evaluate flux integrals, one generalizing FTC and the other generalizing Green's Theorem.

1. Stoke's Theorem

Before we state Stoke's Theorem, we need a definition.

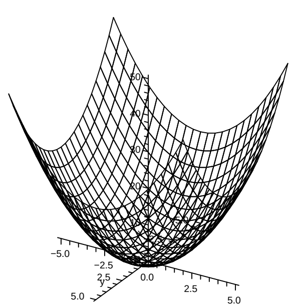
Definition 1.1. Suppose S is an oriented surface with orientation $\vec{n}$ and with boundary B (which is a space curve). Then we define an orientation on B called the induced orientation on B as follows: if we start walking around B standing in the same direction as the orientation $\vec{n}$, then the surface is always on our left.

We look at a couple of examples.

Example 1.2. (*i*) Find the boundary and the orientation of the boundary for the unit sphere if it has outward orientation.

The unit sphere has no boundary (it is called a closed surface), so obviously there is no orientation on it.

(*ii*) Find the boundary and the orientation of the boundary for the surface $z = x^2 + y^2$ over the rectangle $[-5, 5] \times [-5, 5]$ sketched below with downward pointing orientation.



The boundary of the surface is the edge which consists of four connected parabolas. Since it is downard orientation, to guarantee that the surface is always on the left as we walk along the boundary, we need to walk in a clcokwise direction.

We are now ready to state Stoke's Theorem.

Result 1.3. Let S be an oriented piecewise-smooth surface that is bounded by a simple closed piecewise smooth boundary B with induced orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ which contains S. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S \text{curl}\vec{F} \cdot d\vec{S}$$

Before we look at examples, we make a few remarks about Stoke's Theorem.

(*i*) Observe that Stoke's Theorem allows us to change a line integral into a flux integral - this seems like a silly thing to do since flux integrals are generally much more difficult than line integrals and is not really the power of Stoke's Theorem.

(*ii*) The real power of Stoke's Theorem is that, provided a field $\vec{G}$ is a curl field, we can transform a flux integral of G into a line integral - specifically, if $\vec{G} = \text{curl}\vec{F}$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S \vec{G} \cdot d\vec{S}$$

(*iii*) Stoke's theorem is a generalization of the FTC for line integrals. Specifically, FTC of line integrals says if $\vec{F}$ is a conservative field, then a line integral of $\vec{F}$ can be evaluated just using the endpoints of the line (or the boundary of the line). Stoke's Theorem says that if $\vec{F}$ is a curl field, then a flux integral of $\vec{F}$ can be evaluated by converting it into a line integral over the boundary.

We illustrate with some examples.

Example 1.4. (*i*) Explain why Stoke's Theorem cannot be used to turn the flux integral

$$\int\int_S x\vec{i} + y\vec{j} + z\vec{k} d\vec{S}$$

into a line integral over the boundary of S.

Recall that $\text{div}(\text{curl})\vec{F} = 0$. However, $\text{div}(x\vec{i} + y\vec{j} + z\vec{k}) = 3 \neq 0$, so $x\vec{i} + y\vec{j} + z\vec{k}$ cannot be a curl field, so we cannot apply Stoke's Theorem (backwards).

(*ii*) Use Stoke's Theorem to evaluate

$$\int\int_S \text{curl}(yz\vec{i} + xz\vec{j} + xy\vec{k}) \cdot d\vec{S}$$

where S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$ oriented upward.

The boundary B of this region is the intersection of the parabola $z = 9 - x^2 - y^2$ with the plane $z = 5$, so the circle $x^2 + y^2 = 4$ at $z = 4$. Using Stoke's Theorem, we have

$$\int\int_S \text{curl}(yz\vec{i} + xz\vec{j} + xy\vec{k}) \cdot d\vec{S} = \int_B (yz\vec{i} + xz\vec{j} + xy\vec{k}) \cdot d\vec{r}$$

This is a conservative vector field with potential function $f(x, y, z) = xyz$ and B is a closed curve, so

$$\int\int_S \text{curl}(yz\vec{i} + xz\vec{j} + xy\vec{k}) \cdot d\vec{S} = \int_B (yz\vec{i} + xz\vec{j} + xy\vec{k}) \cdot d\vec{r} = 0$$

(iii) Use Stoke's Theorem to evaluate

$$\int_B ((x + y^2)\vec{i} + (y + z^2)\vec{j} + (z + x^2)\vec{k}) \cdot d\vec{r}$$

where B is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ oriented counterclockwise.

We apply Stoke's Theorem directly - we have

$$\text{curl}((x + y^2)\vec{i} + (y + z^2)\vec{j} + (z + x^2)\vec{k})$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (0 - 2z)\vec{i} + (0 - 2x)\vec{j} + (0 - 2y)\vec{k} = -2z\vec{i} - 2x\vec{j} - 2y\vec{k}$$

Therefore, by Stoke's Theorem, we have

$$\int_B ((x+y^2)\vec{i}+(y+z^2)\vec{j}+(z+x^2)\vec{k}) \cdot d\vec{r} = \int\int_S (-2z\vec{i} - 2x\vec{j} - 2y\vec{k}) \cdot d\vec{S}$$

with S oriented upward.

The surface S is a portion of a plane above a region in the xy-plane. To find the equation for the plane, we use two vectors in the plane (displacement vectors between the vertices) and then one of the points and to find the region D, we project into the xy-plane: we get $z = g(x, y) = 1 - x - y$ and D with $0 \leqslant y \leqslant 1 - x$ and $0 \leqslant x \leqslant 1$. Applying the flux integral formula for graphs of functions, we have

$$\int\int_S \vec{F}(x, y, z) \cdot d\vec{S}$$

$$= \int\int_D \left(-P(x, y, g(x, y))\frac{\partial g}{\partial x} - Q(x, y, g(x, y))\frac{\partial g}{\partial y} + R(x, y, g(x, y))\right) dA$$

$$= \int\int_D (-(-2(1-x-y))(-1) - (-2x)(-1) + -2y) dA = \int_0^1 \int_0^{1-x} -2 dy dx$$

$$= -2\int_0^1 (1-x)dx = -2\left[x - \frac{x^2}{2}\right]_0^1 = -1.$$

We finish with some further examples illustrating some important consequences of Stoke's Theorem as well as some indirect applications.

Example 1.5. Suppose that S is a closed surface and $\vec{F}$ is a curl field. Show that

$$\int\int_S \vec{F} \cdot d\vec{S} = 0.$$

Since $\vec{F}$ is a curl field, we must have $\vec{F} = \text{curl}(\vec{G})$ for some vector field $\vec{G}$. This means we can apply Stoke's Theorem indirectly. Specifically, we have

$$\int\int_S \vec{F} \cdot d\vec{S} = \int_B \vec{G} \cdot d\vec{r}$$

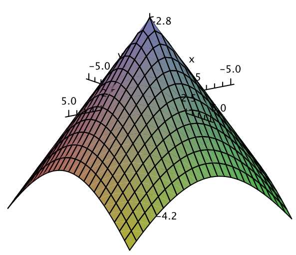
where B is the boundary of S. However, since S is closed, there is no boundary, so it follows that

$$\int\int_S \vec{F} \cdot d\vec{S} = \int_B \vec{G} \cdot d\vec{r} = 0.$$

Example 1.6. Use Stoke's Theorem to evaluate

$$\int\int_S \text{curl}(x^2y^3z\vec{i} + \sin(xyz)\vec{j} + xyz\vec{k}) \cdot d\vec{S}$$

where S is the part of the cone $z = 3 - \sqrt{(x^2+z^2)}$ below the plane $z = 3$ oriented upwards illustrated below:



This looks like a particularly complicated example, but we shall see how in can be turned into a fairly easy problem using Stoke's Theorem. Let $\vec{F}$ denote the vector field given above. First note that since $\vec{F}$ is a curl field, we know that if L is any closed surface, then

$$\int\int_L \vec{F} \cdot d\vec{S} = 0.$$

Let L be the surface which consists of the surface S described above, and the cap of the cone, C oriented upward. Then

$$\int\int_L \vec{F} \cdot d\vec{S} = 0.$$

However,

$$\int\int_L \vec{F} \cdot d\vec{S} = \int\int_S \vec{F} \cdot d\vec{S} + \int\int_C \vec{F} \cdot d\vec{S} = 0,$$

and so

$$\int\int_S \vec{F} \cdot d\vec{S} = -\int\int_C \vec{F} \cdot d\vec{S} = \int\int_{-C} \vec{F} \cdot d\vec{S}$$

where $-C$ denotes the cap of the cone with downward orientation. In particular, we have changed a very complicated problem into a much easier problem. Notice that the cap occurs in the xy-plane, and in this plane, we have $z = 0$. In particular, in this plane, we have $\vec{F} = \vec{0}$ too (by plugging $z = 0$ into the equation). It follows that

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_{-C} \vec{0} \cdot d\vec{S} = 0.$$