The Residue Theorem

"Integration Methods over Closed Curves for Functions with Singularities"

We have shown that if $f(z)$ is analytic inside and on a closed curve C, then

$$\int_C f(z)dz = 0.$$

We have also seen examples where $f(z)$ is analytic on the curve C, but not inside the curve C and

$$\int_C f(z)dz \neq 0$$

(for example $f(z) = 1/z$ over the unit circle centered at 1). In the latter instance however, we had to calculate the integral directly by brute force. In the following few sections, we shall develop methods to integrate functions with singularities over closed curves which avoid direct computation and then we shall use them to solve other related (and seemingly unrelated) problems in math.

1. The Cauchy Residue Theorem

Before we develop integration theory for general functions, we observe the following useful fact.

Proposition 1.1. *Suppose that $f(z)$ has an isolated singularity at z_0 and*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

is its Laurent expansion in a deleted neighbourhood of z_0. Then if C is any circle surrounding z_0 and containing no other isolated singularities and it is oriented counterclockwise, then

$$\int_C f(z)dz = 2\pi i a_{-1}.$$

Proof. By our earlier results, in the Laurent expansion for $f(z)$ around z_0, for a given k we have

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz,$$

so using $k = 1$, the result follows.

□

The coefficient a_{-1} will be very important for our uses so we give it its own name.

Definition 1.2. If

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

in a deleted neighbourhood of z_0, then we call the coefficient a_{-1} the residue of f at z_0 and we denote it by $Res(f; z_0)$.

Evaluation of residues is fairly straight forward and we do not (always) have to find the Laurent expansion explicitly to find residues. Specifically, if we have a pole, we can use the following results.

Proposition 1.3. *Suppose z_0 is a pole of $f(z) = A(z)/B(z)$.*

(i) *If z_0 is a simple pole (of order 1) and $B(z)$ has a simple zero at z_0, then*

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z) = \frac{A(z)}{B'(z)}$$

(ii) *If z_0 is a pole of order k, then*

$$a_{-1} = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dk} (z - z_0)^k f(z)$$

Proof. (i) By assumption, we have

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

so

$$(z - z_0) f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \dots$$

giving

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \dots) = a_{-1}.$$

Alternatively, we have

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) \frac{A(z)}{B(z)} = \lim_{z \to z_0} A(z) \frac{(z - z_0)}{B(z) - B(z_0)}$$

since $B(z_0) = 0$, so we get

$$a_{-1} = \lim_{z \to z_0} A(z) \frac{(z - z_0)}{B(z) - B(z_0)} = \lim_{z \to z_0} \frac{A(z)}{\frac{B(z) - B(z_0)}{z - z_0}} = \frac{A(z_0)}{B'(z_0)}$$

noting that $B'(z_0) \neq 0$ since the pole is of order 1 (using the homework problem).

$\square$

This proposition can be used to evaluate the residue for functions with simple poles very easily and can be used to evaluate the residue for functions with poles of fairly low order. However, it becomes increasingly difficult the higher the order of the pole, and impossible with essential singularities. In these cases, we have no choice but to return to the Laurent expansion.

Example 1.4. Find the residues of $f(z) = \sin(z)/z^2$ and $g(z) = e^{-1/z^2}$ at $z = 0$ and use it to evaluate

$$\int_C f(z)dz$$

and

$$\int_C g(z)dz$$

where C is the unit circle centered at the origin..

(i) We could apply the above results, but first we would need to determine what the order of the pole of $f(z)$ at $z = 0$ is (it looks like a pole of order 2, but recall that $\sin(z)/z$ has a removable singularity at $z = 0$). With this in mind, we instead use the Laurent expansion. We have

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

so

$$\frac{\sin(z)}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \dots$$

so the residue is 1 (and in fact $z = 0$ is a pole of order 1). Using the earlier proposition, we have

$$\int_C f(z)dz = 2\pi i * 1 = 2\pi i.$$

(ii) We have

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

so

$$e^{-1/z^2} = 1 - \frac{1}{z^2} + \frac{1}{2!z^4} - \frac{1}{3!z^6} + \dots$$

so the residue is 0. Using the earlier proposition, we have

$$\int_C f(z)dz = 2\pi i * 0 = 0.$$

By the first proposition we gave, we can use residues to evaluate integrals of functions over circles containing a single. To evaluate general integrals, we need to find a way to generalize to general closed curves which can contain more than one singularity. First we recall some simple facts and definitions about closed curves.

Definition 1.5. A closed curve is called simple if it does not intersect itself (the book calls such curves "regular curves").

Theorem 1.6. *Any closed curve in $\mathbb{C}$ can be written as a union of simple closed curves.*

Proposition 1.7. *If C is a closed curve and $C = C_1 \cup \cdots \cup C_n$ (with the relevant imposed orientations) where the C_i are simple closed curves, then*

$$\int_C f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz$$

Example 1.8. Evaluate

$$\int_C \frac{\sin(z)}{z^2} dz$$

where C is one of the following curves:

(i) We can break up C into two curves, one oriented counterclockwise around the origin - C_1, and the other clockwise - C_2. Next note that since there are no singularites contained between C_1 and a small circle C_3 centered at the origin oriented counterclockwise, and C_2 and C_3, we have

$$\int_{C_1} f(z)dz = \int_{C_3} f(z)dz$$

and

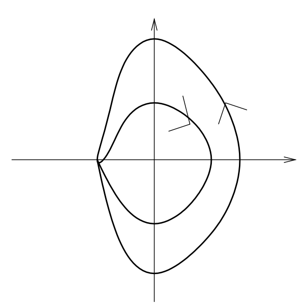
$$\int_{C_2} f(z)dz = -\int_{C_3} f(z)dz$$

Then using our earlier proposition, we have

$$\int_{C_3} f(z)dz = 2\pi i$$

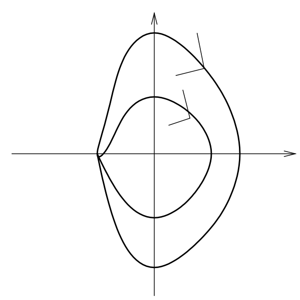
It follows that

$$\int_C f(z)dz = 2\pi i - 2\pi i = 0.$$



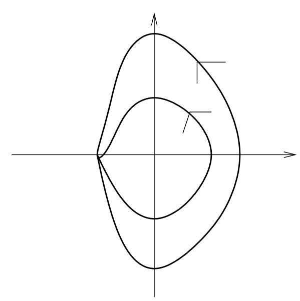
(*ii*) We can break up C into two curves, both oriented clockwise around the origin. Using the same argument as above, it follows that

$$\int_C f(z)dz = -2\pi i - 2\pi i = -4\pi i.$$



(*iii*) We can break up C into two curves, both oriented counterclockwise around the origin. Using the same argument as above, it follows that

$$\int_C f(z)dz = 2\pi i + 2\pi i = 4\pi i.$$



With these results and observing the the examples above, it suffices to determine a formula to integrate a function $f(z)$ over simple closed depending upon its orientation. Before we develop the formula however, we have a couple of necessary definitions.

Definition 1.9. If C is a closed simple curve, we call the compact region bounded by C the "inside" of C.

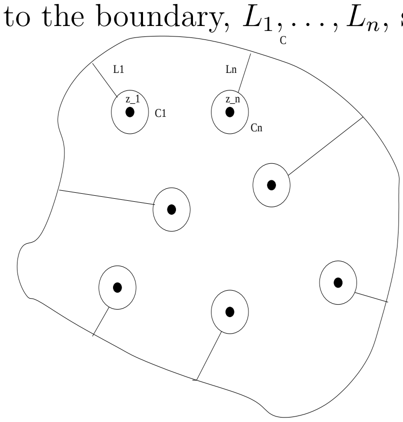
Definition 1.10. We say a simple closed curve C is oriented counterclockwise if as a particle moves around C in the direction of the orientation, the "inside" of C is to the left of the particle.

We are now ready to prove the main result.

Theorem 1.11. *(Cauchy's Residue Theorem) Suppose $f(z)$ is analytic in a simply connected region D except for isolated singularities. Let γ be a simple closed curve in D which does not contain any singularities oriented counterclockwise and suppose the singularities $z_1, \ldots, z_n$ lie in* *the inside of C. Then*

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} Res(f, z_i).$$

Proof. In order to prove this result, we shall use a generalization of the previous result we proved stating that the integral over a closed simple curve is equal to the integral over any closed curve inside provided $f(z)$ is analytic in between the two curves. Specifically, if $z_1, \ldots, z_n$ are the singularities inside C, we can draw small circles around them, $C_1, \ldots, C_n$ which are fully contained in the interior of C and lines connected these circles to the boundary, $L_1, \ldots, L_n$, see illustration.



Starting at the end of L_1 on C, we define the curve K by traversing L_1 to C_1, traversing C_1 counterclockwise, traversing L_1 back toward $C$and then traversing C counterclockwise until the end of L_2 and continuing in this fashion until we traverse the whole of C. As with the earlier proposition, the interior of K is simply connected, and since $z_1, \ldots, z_n$ are the only singulairites of $f(z)$, we get

$$\int_K f(z)dz = 0.$$

Next note that since all the L_i's cancel (since we are traversing each curve in both directions), we get

$$\int_{C \cup -C_1 \cup -C_2 \cup \cdots \cup -C_n} f(z)dz = 0$$

or

$$\int_C f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz.$$

Applying our first result, we get

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} Res(f, z_i).$$

□

Corollary 1.12. *Suppose $f(z)$ is analytic in a simply connected region D except for isolated singularities. Let γ be a simple closed curve in D which does not contain any singularities oriented clockwise and suppose the singularities $z_1, \ldots, z_n$ lie in the inside of C. Then*

$$\int_C f(z)dz = -2\pi i \sum_{i=1}^{n} Res(f, z_i).$$

Proof. This is simply due to the fact that C is oriented in the opposite direction to that given in the previous result. □

2. Application of the Residue Theorem

We shall see that there are some very useful direct applications of the residue theorem. However, before we do this, in this section we shall show that the residue theorem can be used to prove some important further results in complex analysis. We start with a definition.

Definition 2.1. We say f is meromorphic in a domain D if f is analytic in D except possibly isolated singularities.

Theorem 2.2. *Suppose C is a simple closed curve. If f is meromorphic inside and on C and contains no zeros or poles on C and if $Z =$ number of zeros (counted with multiplicity) and $P =$ number of poles (counted with multiplicity), then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = Z - P.$$

Proof. In order to show this, we shall calculate the residues at each of the poles of f'/f in D, the interior of C and then apply the Residue Theorem. First note that f'/f will be analytic at all points except the zeroes and poles of $f(z)$, so we consider these two possibilities.
First, if $z_0 \in D$ is a zero of $f(z)$, of multiplicity k, then $f(z) = (z - z_0)^k g(z)$ for some function $g(z)$ which is analytic and nonzero at z_0. Then we have $f'(z) = k(z - z_0)^{k-1} g(z) + (z - a)^k g'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{k(z - z_0)^{k-1} g(z) + (z - a)^k g'(z)}{(z - z_0)^k g(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}$$

near z_0. Since g'/g is analytic at z_0, it follows that f'/f will have a simple pole at z_0 with residue k.
If $(z - z_0)$ is a pole of $f(z)$ of order k, then $f(z) = g(z)(z - z_0)^{-k}$ for some function $g(z)$ which is analytic and nonzero at z_0. Then we have $f'(z) = -k(z - z_0)^{-k-1} g(z) + (z - a)^{-k} g'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{-k(z - z_0)^{-k-1} g(z) + (z - a)^{-k} g'(z)}{(z - z_0)^{-k} g(z)} = -\frac{k}{z - z_0} + \frac{g'(z)}{g(z)}$$

near z_0. Since g'/g is analytic at z_0, it follows that f'/f will have a simple pole at z_0 with residue $-k$. Applying the residue theorem, the result follows.

$\square$

Theorem 2.3. *(Argument Principle) Suppose $f(z)$ is analytic and nonzero on a closed simple curve C, fix some z_0 on C and let $\Delta_C Arg(f)$ denote the change in argument (measured in total radians of change - not restricted to $0 \leqslant Arg(z) \leqslant 2\pi$) from z_0 when traversing the curve in a counterclockwise direction. Then*

$$\frac{1}{2\pi}\Delta_C Arg(f) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz.$$

Proof. First, if $z(t)$ parameterizes the curve C with $a \leqslant t \leqslant b$, then we have

$$\int_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{f'(z(t))}{f(z(t))}\dot{z}(t)dt.$$

Suppose that $f(z) = r(t)e^{i\vartheta(t)}$. Next, recall (as shown in Chapter 3) that if $w(t) = f(z(t))$, then $w'(t) = f'(z(t))\dot{z}(t)$ (this is simply the chain rule for complex functions depending upon a single parameter). It follows that along smooth arcs making up the curve C, we have

$$f'(z(t))\dot{z}(t) = \frac{d}{dt}f(z(t)) = \frac{d}{dt}(r(t)e^{i\vartheta(t)}) = r'(t)e^{i\vartheta(t)} + ir(t)e^{i\vartheta(t)}\vartheta'(t).$$

Therefore

$$\int_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{f'(z(t))}{f(z(t))}\dot{z}(t)dt = \int_a^b \frac{r'(t)e^{i\vartheta(t)} + ir(t)e^{i\vartheta(t)}\vartheta'(t)}{r(t)e^{i\vartheta}}dt$$

$$= \int_a^b \frac{r'(t)}{r(t)}dt + \int_a^b i\vartheta'(t)dt = \ln\left(r(t)\right)\Big|_1^b + \vartheta(t)\Big|_a^b = 0 + \Delta_C Arg(f)$$

since the first integral is purely real and the second integral is simply the change in argument from $z(a)$ to $z(b)$ (which is measured as the actual change, even though $z(a) = z(b)$).

$\square$

Corollary 2.4. *Suppose C is a simple closed curve. If f is meromorphic inside and on C and contains no zeros or poles on C and if Z = number of zeros (counted with multiplicity) and P = number of poles (counted with multiplicity), and $\Delta_C Arg(f)$ denotes the change in argument when traversing once around C, then*

$$Z - P = \frac{1}{2\pi}\Delta_C Arg(f).$$

The argument principle provides a way to measure the angle change when traversing a closed curve once. Since there are 2π radians in a complete revolution, the value of

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

gives the number of times a curve wraps around the origin (and is consequently sometimes referred to as the winding number of C). Given the corollary, it follows that $Z-P$ gives the winding number of a curve C. A consequence of the argument principle and these observations is the following.

Theorem 2.5. *(Rouchés Theorem) Suppose that f and g are analytic inside and on a closed regular curve C and that $|f(z)| > |g(z)|$ for all $z \in C$. Then $Z(f+g) = Z(f)$ inside C (where $Z(f)$ denotes the number of zeros of f inside C).*

Proof. First, on C observe that $|f(z)+g(z)| \geqslant |f(z)| - |g(z)| > 0$. Therefore, since we are assuming $|f(z)| > |g(z)| \geqslant 0$, it follows that $f(z)$ has no zeros on C and neither does $f(z)+g(z)$. Since both f and $f+g$ are analytic, it follows that the number of zeros of f and $f+g$ inside C are $(\Delta_C Arg(f))/2\pi$ and $(\Delta_C Arg(f+g))/2\pi$ respectively. Next using the basic properties of the argument, we observe that

$$\Delta_C Arg(1+\frac{g}{f}) = \Delta_C Arg(\frac{f+g}{f}) = \Delta_C Arg(f+g) - \Delta_C Arg(f).$$

Therefore it suffices to show that

$$\Delta_C Arg(1+\frac{g}{f}) = 0.$$

Observe however that $|f| > |g|$, so $|g/f| < 1$ and consequently

$$\left|\left(1+\frac{g(z)}{f(z)}\right) - 1\right| < 1.$$

However, this means that all the values of $1+f/g$ take place inside the circle of radius 1 centered at $z=1$, and consequently there is no way the function $1+g/f$ could wrap around the origin. It follows that

$$\Delta_C Arg(1+\frac{g}{f}) = 0$$

and hence $Z(f+g) = Z(f)$.

$\square$

We finish with a couple of examples.

Example 2.6. (i) Show that $z^6 + 3z^4 - 2z + 8$ has all 6 zeros satisfying $1 < |z| < 2$.

We shall attempt to apply Rouch/'es Theorem. First observe that on $|z| = 2$, if $f(z) = z^6$ and $g(z) = 3z^4 - 2z + 8$, then $|f(z)| = 64$ and $|g(z)| \leqslant 3 \cdot 2^3 + 2 \cdot 2 + 8 = 60$, so

$|f(z)| > |g(z)|$. It follows that $f(z) + g(z)$ has the same number of zeros as $f(z)$ for $|z| < 2$, so all 6 zeros of z^6+3z^4-2z+8 occur in $|z| < 2$.

Next observe that for $|z| = 1$, if we take $f(z) = z^6 + 8$ and $g(z) = 3z^4-2z$, then $|f(z)| \geqslant 8-1 = 7$ and $|g(z)| \leqslant 3+2 = 5$, so it follows that $f(z) + g(z)$ has the same number of zeros in $|z| < 1$ as $f(z) = z^6 + 8$. However, all the zeros of $z^6 + 8$ have modulus $8^{(}1/6) > 1$, so none occur in $|z| < 1$. Thus it follows that no zeros of $z^6 + 3z^4 - 2z + 8$ occur in $|z| < 1$. Thus all zeros occur in the annulus $1 < |z| < 2$.

(ii) Show that the quartic polynomial $p(z) = z^4 + z^3 + 1$ has one zero in each quadrant.

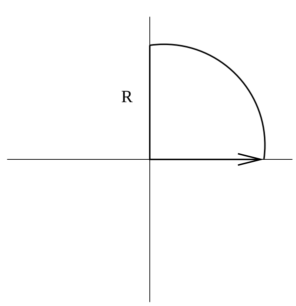
First observe that there is no zero on the real axis (since $x^4+x^3 > -1$ for all real x using elementary calculus). Likewise, there is no zero on the imaginary axis since $(iy^4) + (iy)^3 + 1 = y^4 - iy^3 + 1 = (y^4 + 1) - iy^3$ always has positive real part i.e. $y^4 + 1 > 0$ for all real y. Thus the zeros of $p(z)$ must occur in the quadrants.

Next note that since $p(z)$ has real coefficients, the zeros of $p(z)$ must come in conjugate pairs. Therefore, it suffices to show that exactly one zero occurs in the first quadrant (since its conjugate will appear in the fourth, and the other pair must appear in the second and third).

We shall use the argument principle to count the number of zeros (since $f(z)$ is analytic, the value of

$$\frac{1}{2\pi}\Delta_C Arg(f(z))$$

will count the number of zeros contained in C). Let C be the curve which consists of the quarter circle in the first quadrant centered at $z = 0$ with radius R and the real and imaginary line segments making this a closed curve (see illustration).

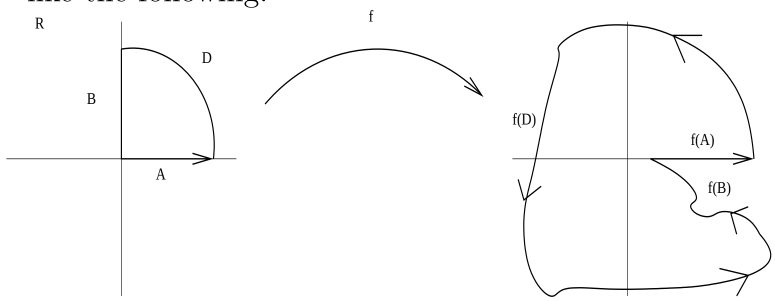


We now consider the image of C under the map $f(z)$. First, we have $f(0) = 1$. Next, the line segment $0 < x < R$ along the real axis (A) maps under $f(z)$ to the curve $x^4 + x^3 + 1$ which is a portion of the positive real axis. The line line segment

$0 < y < R$ along the imaginary axis (B) maps under $f(z)$ to the curve $(y^4+1) - iy^3$ which has strictly positive real part and strictly negative imaginary part. Finally, the quarter circle $Re^{i\vartheta}$ with $0 \leqslant \vartheta \leqslant \pi/2$ (D) maps to

$$R^4 e^{4i\vartheta} + R^3 e^{3i\vartheta} + 1 = R^4 e^{4i\vartheta}\left(1 + \frac{e^{-i\vartheta}}{R} + \frac{e^{-4i\vartheta}}{R^4}\right).$$

It follows that the image of C under $f(z)$ lookes something like the following:



In particular, observe that

$$R^4 e^{4i\vartheta}\left(1 + \frac{e^{-i\vartheta}}{R} + \frac{e^{-4i\vartheta}}{R^4}\right)$$

has argument close to 4ϑ for large values of R, so the image of D under $f(z)$ will have a change in argument of approximately 2π. Next, since A and B both map to curves with strictly real part, they could not wrap around the origin, so the change in ϑ must be 2π. It follows that the number of zeros contained in C for sufficiently large R will be equal to

$$\frac{1}{2\pi}\Delta_C Arg(f(z)) = \frac{1}{2\pi}2\pi = 1.$$

Homework:
Questions from pages 126-127; 1,2,5,6,7,8