Functions of a Complex Variable

"Functions".

In this chapter, we shall examine in more detail the concept of a complex function - that is a function of a complex variable which takes complex values. We shall start by looking at the simplest type of function - a polynomial - and discuss different results from real analysis which can be extended to complex polynomials. This is mainly due to the fact that many of the results we shall prove regarding complex functions will be greatly simplified when only considering complex polynomials. Following this, once we have a thorough understanding of complex polynomials, we shall move on to other types of complex functions.

1. Analytic Polynomials

1.1. Before we study the properties of complex polynomials, we state a formal definition which is simply a direct generalization of the real variable definition.

Definition 1.1. A complex polynomial is an expression of the form

$$P(z) = \sum_{n=0}^{N} a_n z^n$$

where $a_n \in \mathbb{C}$ and z is a complex variable.

Since $z = x + iy$ where x and y are real numbers, any complex polynomial $P(z)$ can be regarded as a polynomial of two real variables $P(x, y)$ with complex coefficients. For example $P(z) = z = x + iy = P(x, y)$. Since we developed a whole theory of calculus for functions of two variables (multivariable calculus), there is an obvious advantage of considering a complex function in this way. This, however leads us to the question of whether every polynomial of two variables with complex coefficients is a polynomial of a complex variable. Specifically, we could ask the following;

Question 1.2. Suppose $P(x, y)$ is a function of two real variables x and y with complex coefficients. Is $P(x, y)$ a complex polynomial - or equivalently, does there exist $Q(z)$ such that $P(x, y) = Q(z)$ for some complex polynomial Q where $z = x + iy$?

Clearly this can only be done if $P(x, y)$ can be factored as a complex variable. We shall be interested especially in such polynomials, so we give them a formal definition.

Definition 1.3. A polynomial $P(x, y)$ will be called an analytic polynomial if there exists complex constants α_k such that

$$P(x, y) = \sum_{n=0}^{N} \alpha_n (x + iy)^n.$$

In this case, P is a complex polynomial of z and we write $P(z)$.

So the question we need to address is how to determine whether or not a given polynomial is analytic.

1.2. Testing whether a Polynomial is Analytic. One method to test for analyticity is by direct brute force.

Example 1.4. (i) Show that $P(x, y) = x^2 + 2ixy - y^2$ is analytic.

We need to show that P can be factored as a complex variable. However, we observe that $P(x, y) = (x + iy)^2 = z^2$, so P is analytic by definition.

(ii) Show that $P(x, y) = x - iy$ is not analytic.

Observe that if P were analytic, then $P(x, y) = \sum_{n=0}^{N} \alpha_n (x + iy)^n$ by definition. If we were to distribute, the highest power of x and y in P would be the N, so in this case, we would have $N = 1$ i.e. $P(x, y) = \alpha_0 + \alpha_1 x + \alpha_1 iy = x - iy$. Clearly there are no solutions to this since we would have $\alpha_0 = 0$ and $\alpha_1 = 1$ or -1. Thus $P(x, y) = x - iy$ is not analytic.

Though in principle this would work for any polynomial, we can see immediately that it will lead to difficult calculations very quickly. Therefore, we need more efficient ways to determine whether a polynomial is analytic. For the special case of polynomials, we have the following nice result.

Proposition 1.5. *A polynomial $P(x, y)$ is analytic if and only if $P(x, y) = P(x + iy, 0)$.*

Proof. First suppose that $P(x, y) = P(x + iy, 0)$. If we write P as two sums, one which contains all products of y, we have:

$$P(x, y) = \sum_{j} \alpha_j x^j + \sum_{l,m} \beta_{l,m} x^l y^m.$$

By assumption,

$$P(x, y) = \sum_{j,k} \alpha_j x^j + \sum_{l,m} \beta_{l,m} x^l y^m = \sum_{j} \alpha_j (x + iy)^j = P(x + iy, 0)$$

so by definition P is analytic.
Now suppose P is analytic. Then

$$P(x, y) = \sum_{n=0}^{N} \alpha_n (x + iy)^n$$

so

$$P(x+iy,0)=\sum_{n=0}^{N}\alpha_n((x+iy)+i*0)^n=\sum_{n=0}^{N}\alpha_n(x+iy)^n=P(x,y).$$

□

This gives a very easy criterion to determine whether a polynomial is analytic, but it does not work for all types of functions.

Example 1.6. Let $P(x,y)=\cos(x)+i\sin(y)$. Then $P(x+iy,0)=\cos(x+iy)$, but we do not even know how to evaluate cosine of a complex number. .

This means we need to determine another criterion for whether a polynomial is analytic which should work for all types of complex functions. We shall do this through the use of partial derivatives which we must first define.

Definition 1.7. Let $f(x,y)=u(x,y)+uv(x,y)$ where u and v are real valued functions of x and y. We define the partial derivatives f_x and f_y of f by $f_x=u_x+iv_x$ and $f_y=u_y+iv_y$ respectively provided they exist.

The main test for analytic polynomials which we shall extend to all functions is the following.

Proposition 1.8. *A polynomial $P(x,y)$ is analytic if and only if $P_y=iP_x$.*

Proof. Suppose $P(x,y)$ is analytic, so

$$P(x,y)=\sum_{n=0}^{N}\alpha_n(x+iy)^n.$$

Then

$$P_x=\sum_{n=0}^{N}n\alpha_n(x+iy)^{n-1}$$

and

$$P_y=\sum_{n=0}^{N}in\alpha_n(x+iy)^{n-1}=i\sum_{n=0}^{N}n\alpha_n(x+iy)^{n-1},$$

so the result follows.
Now suppose that $P_x=iP_y$. To show that P is analytic, it suffices to how that each component of P of some fixed degree is analytic. Therefore, suppose $P(x,y)$ has nth degree terms of the form

$$Q(x,y)=a_0x^n+a_1x^{n-1}y+\cdots+a_ny^n.$$

Since $Q_y=iQ_x$, we have

$$(ina_0x^{n-1}+(n-1)a_1x^{n-2}y+(n-1)a_2x^{n-3}y^2+\cdots+2a_{n-1}xy^{n-1}+a_{n-1}y^{n-1})$$

$$= a_1x^{n-1} + 2a_2x^{n-2}y \cdots + (n-1)a_{n-1}xy^{n-1} + na_ny^{n-1}.$$

Comparing coefficients, all terms can be written in terms of a_0. Specifically, we have

$$a_1 = ina_0,$$
$$a_2 = i\frac{n-1}{2}a_1 = i^2\frac{n(n-1)}{2}a_0,$$
$$a_3 = i\frac{n-2}{3}a_2 = i^3\frac{n(n-1)(n-2)}{2*3}a_0,$$

and in general

$$a_k = i\frac{n-k+1}{k}a_2 = i^k\frac{n!}{(n-k)!k!}a_0 = i^k\binom{n}{k}a_0.$$

Thus we have

$$Q(x,y) = \sum_{k=0}^{n} a_kx^{n-k}y^k = a_0\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(iy)^k = C_0(x+iy)^n$$

using the binomial Theorem. Hence Q is analytic, and so is P.

□

Another way to state this Proposition is to write it in terms of the real parts, $u(x,y)$ and imaginary part, $v(x,y)$ of the polynomial P. Specifically, we have the following.

Proposition 1.9. *A polynomial $P(x,y) = u(x,y)+iv(x,y)$ is analytic if it satisfies the two equations*

$$u_x = v_y; u_y = -v_x$$

called the Cauchy-Riemann equations.

Example 1.10. For $x^2 - y^2 + 2ixy$, we have $P_x = 2x - 2iy$ and $P_y = -2y + 2ix$, so we have $P_y = iP_x$ meaning P is analytic.
For $x - iy$, we have $P_x = 1$ and $P_y = -i$, so $iP_x = i \neq -i = P_y$, so P is not analytic.

1.3. Derivatives. Analytic polynomials share a number of properties with real variable functions, one of the more important being the fact that the derivative can be defined in much the same way, and when realized as a function of z, most of the rules hold. We make our definition for the derivative more general so it holds for functions which are not polynomials.

Definition 1.11. A complex values function f defined in a neighbourhood of z is said to be differentiable at z if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists and we denote it by $f'(z)$.

Note that unlike single variable calculus, we must be careful about direction of approach - the limit must exist and be equal regardless of how $h \to 0$.

Example 1.12. Consider the function $f(z) = \bar{z}$. At any point z, we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h\to 0} \frac{\bar{h}}{h}$$

which is 1 if h is purely real and -1 if h is purely imaginary. In particular, $f(z) = \bar{z}$ is not differentiable anywhere.

There are many results from calculus which generalize to complex functions.

Proposition 1.13. *If f and g are both differentiable at z, then so are $h_1 = f + g$, $h_2 = fg$ and if $g(z) \neq 0$, $h_3 = f/g$ and we have*

$$h_1'(z) = f'(z) + g'(z)$$

$$h_2'(z) = f'(z)g(z) + f(z)g'(z)$$

and

$$h_3'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

Proof. For old times sake, we shall prove the product rule. By assumption,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h},$$

and

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h}$$

exist and we observe that

$$\lim_{h\to 0} f(z+h) = f(z).$$

Then we have

$$\lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h}$$

$$= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z+h)g(z) + f(z+h)g(z) - f(z)g(z)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(z+h)g(z+h) - f(z+h)g(z)}{h} + \frac{f(z+h)g(z) - f(z)g(z)}{h} \right)$$

$$= \lim_{h\to 0} f(z+h) \frac{g(z+h) - g(z)}{h} + \lim_{h\to \infty} g(z) \frac{f(z+h) - f(z)}{h} = f(z)g'(z) + g(z)f'(z).$$

□

Note that the previous result holds for any differentiable functions of z (not just polynomials). The following result, allows us to differentiate complex polynomials in the same way as real polynomials.

Proposition 1.14. *If $P(z) = \sum_{k=0}^{n} \alpha_k z^k$, then $P'(z) = \sum_{k=0}^{n} k\alpha_k z^{k-1}$.*

2. Power Series

Complex polynomials are finite sums of monomials. The obvious generalization is to consider infinite sums of monomials (or power series) and consider which results we have considered generalize to power series. We should note that the ideas of power series will become important later in the course.

2.1. Basic Definitions for Power Series.

We define a power series of a complex variable as follows.

Definition 2.1. A power series in a complex variable z is an infinite series of the form

$$\sum a_k z^k.$$

Power series are formal sums and only really make sense as functions if they converge. In order to discuss convergence of power series, we recall the following definition from real analysis.

Definition 2.2. We define $\overline{\lim}$ called the limsup of a sequence of real numbers $\{a_n\}$ to be the limit

$$\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty}\left(\sup_{k\geqslant n} a_k\right).$$

Suppose We shall need the following facts about $\overline{\lim}$.

(i) for each N and for each $\varepsilon > 0$, there exists $k > N$ such that $a_k \geqslant L - \varepsilon$.

(ii) for each $\varepsilon > 0$, there is some N such that $a_k \leqslant L + \varepsilon$ for all $k > N$.

(iii) $\overline{\lim} ca_n = cL$ for any nonnegative constant c.

With these definitions and facts, we can prove a result which will help determine convergence of a power series. Observe that this is a generalization of determining convergence developed in Calculus 2 (though we provide a proof).

Theorem 2.3. *Suppose* $\overline{\lim}|a_k|^{\frac{1}{k}} = L$.

(i) *If* $L = 0$, $\sum a_k z^k$ *converges for all* z.

(ii) *If* $L = \infty$, $\sum a_k z^k$ *converges for* $z = 0$ *only.*

(iii) *If* $0 < L < \infty$, $\sum a_k z^k$ *set* $R = 1/L$. *Then we call* R *the radius of convergence and* $\sum a_k z^k$ *converges for all* z *with* $|z| < R$ *and diverges for* $|z| > R$. *For* $|z| = R$, *it could either converge or diverge.*

Proof. We prove the three cases separately.

(i) $L = 0$

Since $\overline{\lim}|a_k|^{\frac{1}{k}} = 0$, it follows that $\overline{\lim}|a_k|^{\frac{1}{k}}|z| = 0$ independent of z. Thus for each z there is some N such that $k > N$ implies

$$|a_k z|^{\frac{1}{k}} \leqslant \frac{1}{2}$$

or equivalently

$$|a_k z^k| \leqslant \frac{1}{2^{\frac{1}{k}}}.$$

Since $\sum 1/2^k$ converges, it follows that $\sum |a_k z^k|$ converges, so $\sum a_k z^k$ converges absolutely and hence converges.

(ii) $L = \infty$

First since $\overline{\lim}|a_k|^{\frac{1}{k}} = \infty$, it follows that for any M there exists N such that if $k > N$, then

$$|a_k|^{\frac{1}{k}} > M.$$

It follows that for any $z \neq 0$, there exists N such that for $k > N$,

$$|a_k|^{\frac{1}{k}} \geqslant \frac{1}{|z|},$$

or equivalently,

$$|a_k|^{\frac{1}{k}}|z| \geqslant 1.$$

It follows that for $k > N$, we have

$$|a_k z^k| \geqslant 1$$

which, as a sequence, doesn't converge to 0 and so $\sum a_k z^k$ cannot converge. Clearly if $z = 0$, this series converges.

(iii) $0 < L < \infty$

First assume $|z| < R$ and set

$$\delta = \frac{1 - \frac{|z|}{R}}{2},$$

so we have $|z| = R(1 - 2\delta)$. Then we have

$$\overline{\lim}|a_k|^{\frac{1}{k}}|z| = LR(1 - 2\delta) = (1 - 2\delta).$$

Using property 2 of $\overline{\lim}$ with $\delta = \varepsilon$, for sufficiently large k, we have

$$|a_k|^{\frac{1}{k}}|z| < (1 - \delta)$$

where $0 < \delta < 1$, or equivalently

$$|a_k z|^k < (1 - \delta)^{\frac{1}{k}}.$$

Since $(1 - \delta) < 1$, the sum $\sum (1 - \delta)^k$ converges, and hence $\sum a_k z^k$ converges absolutely and hence it converges.

If $|z| > R$, then

$$\overline{\lim}|a_k|^{\frac{1}{k}}|z| = L|z| > 1.$$

This means for sufficiently large k, we must have

$$|a_k|^{\frac{1}{k}}|z| > 1,$$

or $|a_k z^k| > 1$. In particular, the sequence $|a_k z^k|$ does not converge to 0 and hence $\sum a_k z^k$ cannot converge.

□

Note that if the series $\sum a_k z^k$ has radius of convergence R, then it will converge uniformly on any smaller disc $|z| \leqslant R - \delta$ for $\delta > 0$. Calculation of $L = \overline{\lim} a_k^{1/k}$ is sometimes difficult, so we note the following other way of calculating L (the proof is a homework assignment).

Proposition 2.4. *For any sequence of positive numbers, if*

$$\overline{\lim} \frac{a_{k+1}}{a_k} = L$$

then

$$\overline{\lim} a_k^{\frac{1}{k}} = L.$$

We illustrate with an example.

Example 2.5. Determine where the power series

$$\sum (1+i)^n z^n$$

converges.
Here we have

$$\overline{\lim}(|1+i|^k)^{\frac{1}{k}} = \overline{\lim}|1+i| = \sqrt{2},$$

so the radius of convergence is $R = 1/\sqrt{2}$, so $\sum(1+i)^n z^n$ converges provided $|z| < 1/\sqrt{2}$ (disc of radius $1/\sqrt{2}$ centered at the origin).
On the disc, it also diverges. Specifically, since $|z| = 1/\sqrt{2}$ on the disc, we have

$$|(1+i)^n z^n| = \sqrt{2}\frac{1}{\sqrt{2}} = 1$$

independent of z. In particular, the sequence $(1+i)^n z^n$ does not converge to 0 on the disc and hence the series cannot converge.

We observe the following facts regarding the arithmetic operations on power series.

Proposition 2.6. *Suppose $\sum a_k z^k$ and $\sum b_k z^k$ are both convergent with radii of convergence R_1 and R_2 respectively with $R_1 > R_2$.*

(*i*) *The sum defined as*

$$\sum (a_k + b_k) z^k = \sum a_k z^k + \sum b_k z^k$$

has radius of convergence R_2.

(*ii*) *The product, or Cauchy product, defined as*

$$\sum c_k z^k = \left(\sum a_k z^k\right)\left(\sum b_k z^k\right)$$

where $c_k = \sum a_k b_{n-k}$ *has radius of convergence* R_2*.*

Note that that convergence of the two series is a necessary condition.

Example 2.7. Consider the two series $\sum(-1)^n$ and $\sum(-1)^n + 1$. Clearly neither of these series converge. Formally, if we take the sum we get $\sum(-1)^n + (-1)^{n+1} = \sum 0 = 0$. Observe however that this is not mathematically sound since in order to sum power series, we must assume that they both converge in the first place!!!

3. Differentiability and Uniqueness of Power Series

Our next task is to show that power series are differentiable functions of z within their radii of convergence and the derivative is calculated in the obvious way.

3.1. Differentiating a Power Series. We are ready to prove the main result regarding derivatives of power series.

Theorem 3.1. *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *converges for* $|z| < R$*. Then* $f'(z)$ *exists and* $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ *for* $|z| < 1$*.*

Proof. We shall prove this first for when $R = \infty$ and then for $R < \infty$.

(*i*) First assume $\sum a_n z^n$ converges for all z. We want to show that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

which is equivalent to showing that

$$\lim_{h\to 0} \left(\frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}\right) = 0.$$

Observe that simplifying, we have

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} a_n \frac{(z+h)^n - z^n}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}$$
$$= \sum_{n=2}^{\infty} a_n \left(\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right).$$

Note however, that for a given n,

$$\left|\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right| = \left|\frac{h z^{n-1} + h^2 z^{n-2} + \cdots + h^n}{h} - n z^{n-1}\right|$$
$$= \left|\sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k}\right| \leqslant \sum_{k=2}^{n} \binom{n}{k} |h^{k-1} z^{n-k}|.$$

Provided $|h| \leqslant 1$, we get

$$\sum_{k=2}^{n} \binom{n}{k} |h^{k-1} z^{n-k}| \leqslant |h| \sum_{k=0}^{n} \binom{n}{k} |z|^{n-k} = |h|(|z|+1)^n$$

Thus for a given h, we have

$$\left| \frac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| \leqslant |h| \sum_{n=0}^{\infty} |a_n|(|z|+1)^n.$$

Since $\sum a_n z^n$ converges for any z, it follows that

$$\sum_{n=0}^{\infty} |a_n|(|z|+1)^n$$

converges to some number A, hence for a given h,

$$\left| \frac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| \leqslant |h| \sum_{n=0}^{\infty} |a_n|(|z|+1)^n = Ah$$

Letting $h \to 0$, we get

$$\lim_{h\to 0} \left| \frac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| = 0$$

or

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Thus by definition, $f(z)$ is differentiable and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

(*ii*) Now assume that $0 < R < \infty$. Let $|z| = R - 2\delta$ with $\delta > 0$ and assume $|h| < \delta$ so $|z+h| \leqslant R$. As with the previous case, we have

$$\frac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=2}^{\infty} a_n \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k}.$$

If $z = 0$, then $b_n = h^{n-1}$ and the proof follows easily. Else we need a better bound.

Observe that

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \leqslant n^2 \binom{n}{k-2}$$

for $k \geqslant 2$. Thus, provided $z \neq 0$, we have

$$\sum_{k=2}^{n} \binom{n}{k} |h^{k-1} z^{n-k}| \leqslant \frac{n^2|h|}{|z|^2} \sum_{k=2}^{n} \binom{n}{k-2} |h^{k-2} z^{n-(k-2)}|$$

$$\leqslant \frac{n^2|h|}{|z|^2} \sum_{j=0}^{n} \binom{n}{j} |h^j z^{n-j}|$$

(since we are at most adding two positive terms)

$$= \frac{n^2|h|}{|z|^2}(|z|+|h|)^2 \leqslant \frac{n^2|h|}{|z|^2}(R-\delta)^n$$

(since $|z|+|h| \leqslant R-2\delta+\delta$). Thus

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} na_n z^{n-1}\right| \leqslant \frac{|h|}{|z|^2}\sum_{n=0}^{\infty} n^2|a_n|(R-\delta)^n \leqslant A|h|$$

for some fixed A since $z \neq 0$ and since $\sum n^2 a_n z^n$ converges for $|z| < R$ (WHY?). Letting $h \to 0$, we conclude that $f'(z)$ exists and $f'(z) = \sum na_n z^{n-1}$.

□

The following corollaries are immediate.

Corollary 3.2. *Power series are infinitely differentiable within their domain of convergence.*

Proof. We prove by induction for

$$f(z) = \sum a_n z^n.$$

The case $n = 1$ is proved above, so assuming for $n = k$, we prove for $n = k+1$. By assumption, we know that $f^k(z)$ exists and

$$f^k(z) = \sum_{n=0}^{\infty} n(n-1)(n-2)\dots(n-k+1)a_n z^{n-k}.$$

This is a power series, so we can calculate its radius of convergence. Observe that

$$\lim_{n\to\infty}(n(n-1)\dots(n-k+1)a_n)^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}}(n-1)^{\frac{1}{n}}\dots(n-k+1)^{\frac{1}{n}}a_n^{\frac{1}{n}}$$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}},$$

so it has the same radius of convergence as

$$\sum a_n z^n.$$

Therefore, by the previous result, $f^{k+1}(z)$ exists on the domain of convergence of

$$\sum a_n z^n.$$

Thus by induction,

□

Corollary 3.3. *If*

$$f(z) = \sum a_n z^n$$

has a nonzero radius of convergence, then

$$a_k = \frac{f^k(0)}{k!}$$

for all k.

Proof. Taking the kth derivative which exists by assumption, we have

$$f^k(z) = \sum_{n=0}^{\infty} n(n-1)(n-2)\dots(n-k+1)a_n z^{n-k}.$$

Pluggin in $z = 0$, we get $f^k(0) = k(k-1)(k-2)\dots 2a_k$, so

$$a_k = \frac{f^k(z)}{k!}.$$

□

3.2. Uniqueness of a Power Series. Next we show that different power series always define different functions of z. Before we do this however, we need some preliminary results.

Proposition 3.4. *If a power series is equal to* 0 *in a neighbourhood of the origin, then it is identically* 0*.*

Proof. Write

$$f(z) = \sum a_n z^n$$

as

$$f(z) = \sum \frac{f^n(z)}{n!} z^n.$$

Since we are assuming this series is zero in a neighbourhood of the origin, it follows that all its derivatives are zero (since the function is not changing). It follows that all the coefficients of the power series are zero and thus the power series is identically 0.

□

In fact, we can prove a much stronger result than this.

Proposition 3.5. *Suppose*

$$\sum a_n z^n$$

is zero at all points of a sequence $\{z_k\}$ which converges to 0*. Then the power series is identically zero.*

Proof. Since $f(z)$ is differentiable, it must be continuous. Thus, by continuity at the origin, we have

$$C_0 = \lim_{z \to 0} f(z) = \lim_{k \to \infty} f(z_k) = 0.$$

But then the function $g(z) = f(z)/z$ is also continuous at the origin (it has the same radius of convergence as $f(z)$ so is continuous at the same points WHY?), and

$$C_1 = g(0) = \lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{k\to\infty} \frac{f(z_k)}{z_k} = 0.$$

Continuing, we get in general that $g_i(z) = f(z)/z^i$ is continuous at the origin, and

$$C_i = g_i(0) = \lim_{z\to 0} g_i(z) = \lim_{z\to 0} \frac{f(z)}{z^i} = \lim_{k\to\infty} \frac{f(z_k)}{z_k^i} = 0.$$

□

Thus we get the following results regarding uniqueness of power series.

Corollary 3.6. *If a power series equals zero at all points of a set with an accumulation point at the origin, the power series is identically zero.*

Proof. We observe that if there is an accumulation point at the origin, there must be some sequence which is a subset and which converges to 0. We then apply the last result.

□

Corollary 3.7. *If*

$$\sum a_n z^n$$

and

$$\sum b_n z^n$$

converge and agree on a set of points with an accumulation point at the origin, then $a_n = b_n$ for all n

Proof. We apply the last Corollary to the power series

$$\sum (a_n - b_n) z^n.$$

□

We note that though all the results we proved were for power series centered around $z = 0$, all results (some slightly modified) will hold for power series centered around an arbitrary complex number $z = \alpha$ i.e. power series of the form

$$\sum a_n (z - \alpha)^n.$$

(*i*) From the book, pages 31-33: Questions 2,3,6,7,9,11,12,16,17,18,19,20