Analytic Functions

"Differentiable Functions of a Complex Variable"

In this chapter, we shall generalize the ideas for polynomials and power series of a complex variable we developed in the previous chapter to general functions of a complex variable. Once we have proved results to determine whether or not a function is analytic, we shall then consider generalizations of some of the more common single variable functions which are not polynomials - namely trigonometric functions and exponential functions.

1. Analyticity and the Cauchy-Riemann Equations

Recall that for a polynomial $P(x, y) = u(x, y) + iv(x, y)$ with complex coefficients, we showed that it was analytic, or differentiable as a function of a complex variable if and only if $P_y = iP_x$, or equivalently if it satisfied the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. In this section, we shall show that this result can be partially extended to any function of a complex variable.

1.1. Determining whether a Function is Analytic.

First we shall show that if a function of a complex variable is differentiable, then it must satisfy the Cauchy-Riemann equations (so it is a necessary condition to satisfy CR).

Proposition 1.1. *If $f = u + iv$ is differentiable at z, then f_x and f_y exist and satisfy the CR equations i.e.*

$$f_y = if_x$$

or

$$u_x = v_y; u_y = -v_x.$$

Proof. Before we prove the result, we make a couple of observations. First note that since we are assuming that f is differentiable,

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists regardless of the way h approaches 0. Secondly, we observe that by definition,

$$f_x = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and

$$f_y = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Now if we take h to be real, then we have

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{f(x+iy+h)-f(x,y)}{h}$$

$$=\lim_{h\to 0}\frac{f((x+h)+iy)-f(x,y)}{h}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=f_x(z).$$

Likewise, taking $h=i\eta$ purely imaginary, we have

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{ih}=\lim_{\eta\to 0}\frac{f(x,y+\eta)-f(x,y)}{i\eta}=\frac{f_y}{i}.$$

Since the limit

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists independent of direction, it follows that $f_x=f_y/i$ or $f_y=if_x$. The CR equations follow.

$\square$

Unfortunately, just because the CR equations exist does not mean that a function is differentiable as the following example shows.

Example 1.2. Show that the function

$$f(x,y)=\sqrt{|xy|}$$

satisfies the CR equations but is not differentiable at $(0,0)$.
Here we have

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

for h real, and

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

for h imaginary, so $0=f_y(0,0)=if_x=0$. However, if we approach along the line $y=x$, we have

$$\lim_{h\to 0}\frac{f(0+h(1+i))-f(0,0)}{h(1+i)}=\lim_{h\to 0}\frac{\sqrt{h^2}-0}{h(1+i)}=\pm\frac{1}{1+i}$$

for h real. In particular, this limit doesn't exist and so f is not differentiable.

As the last example illustrates, Satisfying the CR equations are not a sufficient condition for analyticity (unlike with polynomials and power series). Under certain stronger assumptions however, the CR equations are enough.

Proposition 1.3. *Suppose f_x and f_y exist in a neighbourhood of z. Then if f_x and f_y are continuous at z and $f_y(z)=if_x(z)$, then f is differentiable at z.*

Proof. We need to show that the limit

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists independent of how h approaches 0. We shall do this by showing that

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=f_x(z)=u_x(z)+iv_x(z)$$

where $f=u(x,y)+iv(x,y)$.
Let $h=\zeta+i\eta$ where ζ and η are real numbers. We consider $u(x,y)$ and $v(x,y)$ separately. Observe that

$$\frac{u(z+h)-u(z)}{h}=\frac{u(x+\zeta,y+\eta)-u(x,y)}{\zeta+i\eta}$$

$$=\frac{1}{\zeta+i\eta}([u(x+\zeta,y+\eta)-u(x+\zeta,y)]+[u(x+\zeta,y)-u(x,y)]).$$

Note that $u(x+\zeta,y+\eta)-u(x+\zeta,y)$ is the change of the single variable function $f(y)=u(x+\zeta,y)$ over the interval $[y,y+\eta]$. Therefore, since f_y exists and is continuous (by assumption), so is u_y, so the mean value theorem for single real variable functions implies there exists some number, say $y+\vartheta_1\eta$ (for some $0\leqslant\vartheta_1\leqslant 1$) in the interval $[y,y+\eta]$ such that

$$u(x+\zeta,y+\eta)-u(x+\zeta,y)=((y+\eta)-y)u_y(x+\zeta,y+\vartheta_1\eta).$$

We get similar results for $[u(x+\zeta,y)-u(x,y)])$, and for the components of the function $v(x,y)$ giving

$$\frac{u(z+h)-u(z)}{h}=\frac{\eta}{\zeta+i\eta}u_y(x+\zeta,y+\vartheta_1\eta)+\frac{\zeta}{\zeta+i\eta}u_x(x+\vartheta_2\zeta,y)$$

and

$$\frac{v(z+h)-v(z)}{h}=\frac{\eta}{\zeta+i\eta}v_y(x+\zeta,y+\vartheta_3\eta)+\frac{\zeta}{\zeta+i\eta}v_x(x+\vartheta_4\zeta,y).$$

Therefore, we get

$$\frac{f(z+h)-f(z)}{h}=\frac{\eta}{\zeta+i\eta}\Big(u_y(z_1)+iv_y(z_2)\Big)+\frac{\zeta}{\zeta+i\eta}\Big(u_x(z_3)+iv_x(z_4)\Big)$$

where $z_k\to z$ for $k=1,2,3,4$ as $h\to 0$.
Next we observe that since $if_x=f_y$, we have

$$f_x=\frac{\eta if_x+\zeta f_x}{\zeta+i\eta}=\frac{\eta}{\zeta+i\eta}f_y+\frac{\zeta}{\zeta+i\eta}f_x.$$

Using these equalities, subtracting f_x from

$$\frac{f(z+h)-f(z)}{h}$$

we get

$$\frac{f(z+h)-f(z)}{h} - f_x(z)$$

$$= \frac{\eta}{\zeta+i\eta}\Big(u_y(z_1)+iv_y(z_2)\Big)+\frac{\zeta}{\zeta+i\eta}\Big(u_x(z_3)+iv_x(z_4)\Big)-\Big(\frac{\eta}{\zeta+i\eta}f_y+\frac{\zeta}{\zeta+i\eta}f_x\Big)$$

$$= \frac{\eta}{\zeta+i\eta}\Big(u_y(z_1)+iv_y(z_2)-f_y(z)\Big)+\frac{\zeta}{\zeta+i\eta}\Big(u_x(z_3)+iv_x(z_4)-f_x(z)\Big).$$

Now note that

$$|\frac{\eta}{\zeta+i\eta}|, |\frac{\zeta}{\zeta+i\eta}| \leqslant 1$$

for all h and

$$u_y(z_1)+iv_y(z_2)-f_y(z), u_x(z_3)+iv_x(z_4)-f_x(z) \to 0$$

as $h \to 0$, so it follows that

$$\frac{f(z+h)-f(z)}{h} - f_x(z) \to 0$$

or

$$\frac{f(z+h)-f(z)}{h} \to f_x(z).$$

Thus

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_x(z),$$

and in particular, $f(z)$ is differentiable at z.

□

Usually we consider a function to be differentiable if it is differentiable in some interval (we would not usually consider a function to be differentiable *only at a point*, see example below).

Example 1.4. Consider $f(x, y) = x^2 + y^2$. Here we have $f_x = 2x$ and $f_y = 2y$, so both partial derivatives are continuous. By the last result, this means f_x is differentiable everywhere the CR equations are satisfied, so just at the point $(0, 0)$.

For this reason, when we define analyticity of a function at a point, rather than just requiring differentiability at a point, we want it to describe the local behaviour at and near the point. Rg

Definition 1.5. We say f is analytic at z if f is differentiable in a neighbourhood of z. Similarly, f is analytic on a set S if it is differentiable on some open set containing S.

Definition 1.6. We call a function which is differentiable everywhere an **entire** function.

1.2. **Generalizing Results from Real Variable Calculus.** Now we have a formal definition for an analytic function, we shall consider a small number of results which we can generalized from their real counterparts. We start by considering how to differentiate an inverse function of a complex analytic function. In order to do this, we need the following definition.

Definition 1.7. Suppose that S and T are open sets and that f is $1-1$ on S with $F(S)=T$. g is called the inverse function of f on T if $f(g(z))=z$ for $z \in T$. g is said to be the inverse of f at a point z_0 if it is the inverse in some neighbourhood of z_0.

Proposition 1.8. *Suppose that g is the inverse of f at z_0 and that g is continuous at z_0. If f is differentiable at $g(z_0)$ and if $f'(g(z_0)) \neq 0$, then g is differentiable at z_0 and*

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

Proof. For any $z \neq z_0$ in a neighbourhood of z_0, we have

$$\frac{g(z)-g(z_0)}{z-z_0} = \frac{1}{\frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0)}}.$$

Since g is continuous at z_0, $g(z) \to g(z_0)$ as $z \to z_0$, so by the differentiability of f

$$\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \frac{1}{f'(g(z_0))},$$

since by definition

$$f'(g(z_0)) = \lim_{z\to z_0} \frac{f(g(z))-f(g(z_0))}{z-z_0}.$$

□

Other results which generalize from single variable are the following.

Proposition 1.9. *If $f = u+iv$ is analytic in a region and u is constant, then f is constant.*

Proof. Since u is constant, $u_x = u_y = 0$. Since f is analytic, it satisfies the CR equations, so it follows that $v_x = v_y = 0$. Applying Proposition 3.17, it follows that u and v are constant and hence f is constant. □

Proposition 1.10. *If f is analytic in a region D and if $|f|$ is constant on D, then f is constant on D.*

Proof. If $|f|=0$, the proof is obvious. Else we have

$$u^2+v^2 = C \neq 0.$$

Taking partial derivatives, we have

$$uu_x + vv_x = 0$$

and

$$uu_y + vv_y = 0$$

(using the chain rule). Using the Cauchy Riemann equations, this can be modified to get

$$uu_x - vu_y = 0$$

and

$$vu_x + uu_y = 0$$

which is equivalent to

$$u^2u_x - uvu_y = 0$$

and

$$v^2u_x + vuu_y = 0.$$

Taking the difference, we get

$$(u^2 + v^2)u_x = 0$$

and since $u^2 + v^2 \neq 0$, it follows that $u_x = v_y = 0$. We get a similar result for u_y and v_x, so the result follows.

□

2. Generalizing Functions from Real Variables

We now consider generalizing some of the functions we know from real variable calculus (by generalizing, we mean that we define a function of a complex variable which agrees with the original real function when evaluated at purely real numbers).

2.1. **A Complex Exponential Function.** We want to define a generalization of the real exponential function to complex variables. Specifically, we want to define a function $f(z)$ satisfying

- (i) $f(z_1 + z_2) = f(z_1)f(z_2)$ for any $z_1, z_2 \in \mathbb{C}$
- (ii) $f(x) = e^x$ for any real x.
- (i) From the book, pages 41-42: Questions

Suppose that $f(z)$ is a function which satisfies these two conditions. Then it follows that

$$f(z) = f(x + iy) = f(x)f(iy) = e^x f(iy),$$

so we need to determine what conditions will be imposed on the purely imaginary part of a complex number. Therefore, suppose that $f(iy) = A(y) + iB(y)$, so we have $f(x + iy) = e^x A(y)ie^x B(y)$. In order for f to be analytic, it needs to satisfy the Cauchy Riemann equations, so we must have $u_x = v_y$ and $u_y = -v_x$ or

$$e^x A(y) = e^x B'(y)$$

and

$$e^x A'(y) = e^x B(y).$$

This means that

$$A''(y) = -A(y).$$

A general solution to this differential equation is

$$A(y) = \alpha \cos(y) + \beta \sin(y)$$

for some real numbers α and β and

$$B(y) = -A'(y) = -\beta \cos(y) + \alpha \sin(y),$$

so we get

$$\begin{aligned} f(z) &= e^x(\alpha \cos(y) + \beta \sin(y)) + ie^x(-\beta \cos(y) + \alpha \sin(y)) \\ &= \alpha e^x(\cos(y) + i \sin(y)) + \beta e^x(\sin(y) - i \cos(y)). \end{aligned}$$

If $f(z)$ agrees with e^x for real numbers, we must have

$$1 = f(0) = \alpha - i\beta,$$

so $\alpha = 1$ and $\beta = 0$. Putting all this together, we have the following:

Definition 2.1. We define the complex exponential function $f(z) = e^z$ as

$$f(z) = f(x + iy) = e^x(\cos(y) + i \sin(y)).$$

Proposition 2.2. *The exponential function satisfies the following:*

(*i*) $|e^z| = e^x$
(*ii*) $e^z \neq 0$ *for any value of* z
(*iii*) $e^{iy} = cis(y)$
(*iv*) $e^z = \alpha$ *has infinitely many solutions for* $\alpha \neq 0$.
(*v*) $(e^z)' = e^z$

Proof. Most of these results are fairly trivial to prove.

(*i*)
$$|e^z| = e^x|\cos(y) + i \sin(y))| = e^x(\cos^2(y) + \sin^2(y)) = e^x.$$

(*ii*) This simply follows because $e^x \neq 0$ for any x and $\cos(y) + i \sin(y)) \neq 0$ for any y.

(*iii*)
$$e^z = e^{x+iy} = x^x e^{iy} = e^x(\cos(y) + i \sin(y))$$
so $e^{iy} = \cos(y) + i \sin(y))$.

(*iv*) Suppose α is some non-zero complex number. Then in polar form, we have $\alpha = re^{i\vartheta} = rcis(\vartheta)$ for some $r > 0$ and some angle $0 \leqslant \vartheta < 2\pi$. It follows that solutions to $e^z = e^x e^{iy} = \alpha$ will be $x = \ln(r)$ and $y = \vartheta + 2k\pi$ for any integer k.

(*v*) Recall that $f'(z) = f_x(z)$, so $(e^z)' = (e^x e^{iy})' = e^x e^{iy} = e^z$.

$\square$

2.2. **Defining the Trigonometric Functions.** Now we have defined the exponential function, we can define trigonometric functions. First we observe that for real values of y, we have

$$e^{iy} = \cos(y) + i\sin(y))$$

and

$$e^{-iy} = \cos(y) - i\sin(y)).$$

Thus for real values of y, we can define

$$\sin(y) = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right)$$

and

$$\cos(y) = \frac{1}{2}\left(e^{iy} + e^{-iy}\right).$$

To define complex trigonometric, we simply extend these definitions to complex variables. Specifically, we define them as follows:

Definition 2.3. We define the complex functions sin and cos as

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

and

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

The complex trig functions share many of the identities with their real counterparts. We illustrate with an example.

Example 2.4. We shall show that the indentity $\cos^2(z) + \sin^2(z) = 1$ holds for complex numbers.
Using the definitions, we have

$$\begin{aligned}\cos^2(z) + \sin^2(z) &= \left[\frac{1}{2}\left(e^{iz} + e^{-iz}\right)\right]^2 + \left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right]^2 \\ &= \frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz}\right) - \frac{1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right) = 1\end{aligned}$$

Of course, they also have many differences (most notably, complex trig functions are not bounded).

Homework: (Pages 41 & 42) 2,4,6,8,11,13,14,16,17,19,21