Analytic Functions

“Differentiable Functions of a Complex Variable”

In this chapter, we shall generalize the ideas for polynomials and power series of a complex variable we developed in the previous chapter to general functions of a complex variable. Once we have proved results to determine whether or not a function is analytic, we shall then consider generalizations of some of the more common single variable functions which are not polynomials - namely trigonometric functions and exponential functions.

1. Analyticity and the Cauchy-Riemann Equations

Recall that for a polynomial \( P(x, y) = u(x, y) + iv(x, y) \) with complex coefficients, we showed that it was analytic, or differentiable as a function of a complex variable if and only if \( P_y = iP_x \), or equivalently if it satisfied the Cauchy-Riemann equations \( u_x = v_y \) and \( u_y = -v_x \). In this section, we shall show that this result can be partially extended to any function of a complex variable.

1.1. Determining whether a Function is Analytic. First we shall show that if a function of a complex variable is differentiable, then it must satisfy the Cauchy-Riemann equations (so it is a necessary condition to satisfy CR).

**Proposition 1.1.** If \( f = u + iv \) is differentiable at \( z \), then \( f_x \) and \( f_y \) exist and satisfy the CR equations i.e.

\[
f_y = if_x
\]

or

\[
u_x = v_y; \ u_y = -v_x.
\]

**Proof.** Before we prove the result, we make a couple of observations. First note that since we are assuming that \( f \) is differentiable,

\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]

exists regardless of the way \( h \) approaches 0. Secondly, we observe that by definition,

\[
f_x = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]

and

\[
f_y = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.
\]
Now if we take \( h \) to be real, then we have
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{f(x + iy + h) - f(x, y)}{h} = f_x(z).
\]
Likewise, taking \( h = i\eta \) purely imaginary, we have
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{ih} = \lim_{\eta \to 0} \frac{f(x, y + \eta) - f(x, y)}{i\eta} = \frac{f_y}{i}.
\]
Since the limit
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]
does not exist independent of direction, it follows that \( f_x = f_y/i \) or \( f_y = if_x \). The CR equations follow.

Unfortunately, just because the CR equations exist does not mean that a function is differentiable as the following example shows.

**Example 1.2.** Show that the function
\[
f(x, y) = \sqrt{|xy|}
\]
satisfies the CR equations but is not differentiable at \((0, 0)\).

Here we have
\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]
for \( h \) real, and
\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]
for \( h \) imaginary, so \( 0 = f_y(0, 0) = if_x = 0 \). However, if we approach along the line \( y = x \), we have
\[
\lim_{h \to 0} \frac{f(0 + h(1 + i)) - f(0, 0)}{h(1 + i)} = \lim_{h \to 0} \frac{\sqrt{h^2} - 0}{h(1 + i)} = \pm \frac{1}{1 + i}
\]
for \( h \) real. In particular, this limit doesn’t exist and so \( f \) is not differentiable.

As the last example illustrates, Satisfying the CR equations are not a sufficient condition for analyticity (unlike with polynomials and power series). Under certain stronger assumptions however, the CR equations are enough.

**Proposition 1.3.** Suppose \( f_x \) and \( f_y \) exist in a neighbourhood of \( z \).
Then if \( f_x \) and \( f_y \) are continuous at \( z \) and \( f_y(z) = if_x(z) \), then \( f \) is differentiable at \( z \).
Proof. We need to show that the limit
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]
exists independent of how \(h\) approaches 0. We shall do this by showing that
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f_x(z) = u_x(z) + iv_x(z)
\]
where \(f = u(x, y) + iv(x, y)\).
Let \(h = \zeta + i\eta\) where \(\zeta\) and \(\eta\) are real numbers. We consider \(u(x, y)\) and \(v(x, y)\) separately. Observe that
\[
u(z + h) - u(z) \over h = \frac{u(x + \zeta, y + \eta) - u(x, y)}{\zeta + i\eta}
\]
Note that \(u(x + \zeta, y + \eta) - u(x + \zeta, y)\) is the change of the single variable function \(f(y) = u(x + \zeta, y)\) over the interval \([y, y + \eta]\). Therefore, since \(f_y\) exists and is continuous (by assumption), so is \(u_y\), so the mean value theorem for single real variable functions implies there exists some number, say \(y + \vartheta_1 \eta\) (for some \(0 \leq \vartheta_1 \leq 1\)) in the interval \([y, y + \eta]\) such that
\[u(x + \zeta, y + \eta) - u(x + \zeta, y) = ((y + \eta) - y)u_y(x + \zeta, y + \vartheta_1 \eta).
\]
We get similar results for \([u(x + \zeta, y) - u(x, y)]\), and for the components of the function \(v(x, y)\) giving
\[v(z + h) - v(z) \over h = \frac{v(x + \zeta, y + \vartheta_3 \eta) + \zeta}{\zeta + i\eta}v_x(x + \vartheta_4 \zeta, y).
\]
Therefore, we get
\[
f(z + h) - f(z) \over h = \eta \over \zeta + i\eta \left(u_y(z_1) + iv_y(z_2)\right) + \zeta \over \zeta + i\eta \left(u_x(z_3) + iv_x(z_4)\right)
\]
where \(z_k \to z\) for \(k = 1, 2, 3, 4\) as \(h \to 0\).
Next we observe that since \(if_x = f_y\), we have
\[
f_x = \frac{\eta f_x + \zeta f_x}{\zeta + i\eta} = \frac{\eta}{\zeta + i\eta} f_y + \frac{\zeta}{\zeta + i\eta} f_x.
\]
Using these equalities, subtracting \(f_x\) from
\[
f(z + h) - f(z) \over h
\]
we get
\[
\frac{f(z+h) - f(z)}{h} - f_x(z)
\]
\[
= \frac{\eta}{\zeta + i\eta} \left( u_y(z_1) + Iv_y(z_2) \right) + \frac{\zeta}{\zeta + i\eta} \left( u_x(z_3) + Iv_x(z_4) \right) - \left( \frac{\eta}{\zeta + i\eta} f_y(z) + \frac{\zeta}{\zeta + i\eta} f_x(z) \right)
\]
\[
= \frac{\eta}{\zeta + i\eta} \left( u_y(z_1) + iv_y(z_2) - f_y(z) \right) + \frac{\zeta}{\zeta + i\eta} \left( u_x(z_3) + iv_x(z_4) - f_x(z) \right).
\]
Now note that
\[
|\frac{\eta}{\zeta + i\eta}|, |\frac{\zeta}{\zeta + i\eta}| \leq 1
\]
for all \(h\) and
\[
u_y(z_1) + iv_y(z_2) - f_y(z), \quad u_x(z_3) + iv_x(z_4) - f_y(z) \to 0
\]
as \(h \to 0\), so it follows that
\[
\frac{f(z+h) - f(z)}{h} - f_x(z) \to 0
\]
or
\[
\frac{f(z+h) - f(z)}{h} \to f_x(z).
\]
Thus
\[
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f_x(z),
\]
and in particular, \(f(z)\) is differentiable at \(z\). \(\square\)

Usually we consider a function to be differentiable if it is differentiable in some interval (we would not usually consider a function to be differentiable only at a point, see example below).

**Example 1.4.** Consider \(f(x, y) = x^2 + y^2\). Here we have \(f_x = 2x\) and \(f_y = 2y\), so both partial derivatives are continuous. By the last result, this means \(f_x\) is differentiable everywhere the CR equations are satisfied, so just at the point \((0, 0)\).

For this reason, when we define analyticity of a function at a point, rather than just requiring differentiability at a point, we want it to describe the local behaviour at and near the point.

**Definition 1.5.** We say \(f\) is analytic at \(z\) if \(f\) is differentiable in a neighbourhood of \(z\). Similarly, \(f\) is analytic on a set \(S\) if it is differentiable on some open set containing \(S\).

**Definition 1.6.** We call a function which is differentiable everywhere an entire function.
1.2. Generalizing Results from Real Variable Calculus. Now we have a formal definition for an analytic function, we shall consider a small number of results which we can generalized from their real counterparts. We start by considering how to differentiate an inverse function of a complex analytic function. In order to do this, we need the following definition.

**Definition 1.7.** Suppose that $S$ and $T$ are open sets and that $f$ is $1-1$ on $S$ with $F(S) = T$. $g$ is called the inverse function of $f$ on $T$ if $f(g(z)) = z$ for $z \in T$. $g$ is said to be the inverse of $f$ at a point $z_0$ if it is the inverse in some neighbourhood of $z_0$.

**Proposition 1.8.** Suppose that $g$ is the inverse of $f$ at $z_0$ and that $g$ is continuous at $z_0$. If $f$ is differentiable at $g(z_0)$ and if $f'(g(z_0)) \neq 0$, then $g$ is differentiable at $z_0$ and

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

**Proof.** For any $z \neq z_0$ in a neighbourhood of $z_0$, we have

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{1}{f'(g(z_0))}.$$

Since $g$ is continuous at $z_0$, $g(z) \to g(z_0)$ as $z \to z_0$, so by the differentiability of $f$

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \frac{1}{f'(g(z_0))},$$

since by definition

$$f'(g(z_0)) = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{z - z_0}.$$

□

Other results which generalize from single variable are the following.

**Proposition 1.9.** If $f = u + iv$ is analytic in a region and $u$ is constant, then $f$ is constant.

**Proof.** Since $u$ is constant, $u_x = u_y = 0$. Since $f$ is analytic, it satisfies the CR equations, so it follows that $v_x = v_y = 0$. Applying Proposition 3.17, it follows that $u$ and $v$ are constant and hence $f$ is constant. □

**Proposition 1.10.** If $f$ is analytic in a region $D$ and if $|f|$ is constant on $D$, then $f$ is constant on $D$.

**Proof.** If $|f| = 0$, the proof is obvious. Else we have

$$u^2 + v^2 = C \neq 0.$$

Taking partial derivatives, we have

$$uu_x + vv_x = 0.$$
and 
\[ uu_y + vv_y = 0 \]
(using the chain rule). Using the Cauchy Riemann equations, this can be modified to get
\[ uu_x - vu_y = 0 \]
and
\[ vu_x + uu_y = 0 \]
which is equivalent to
\[ u^2 u_x - uvu_y = 0 \]
and
\[ v^2 u_x + vv u_y = 0. \]
Taking the difference, we get
\[ (u^2 + v^2)u_x = 0 \]
and since \( u^2 + v^2 \neq 0 \), it follows that \( u_x = v_y = 0 \). We get a similar result for \( u_y \) and \( v_x \), so the result follows.

\[ \square \]

2. GENERALIZING FUNCTIONS FROM REAL VARIABLES

We now consider generalizing some of the functions we know from real variable calculus (by generalizing, we mean that we define a function of a complex variable which agrees with the original real function when evaluated at purely real numbers).

2.1. A Complex Exponential Function. We want to define a generalization of the real exponential function to complex variables. Specifically, we want to define a function \( f(z) \) satisfying
\[
\begin{align*}
(i) \quad f(z_1 + z_2) &= f(z_1)f(z_2) \text{ for any } z_1, z_2 \in \mathbb{C} \\
(ii) \quad f(x) &= e^x \text{ for any real } x.
\end{align*}
\]
(i) From the book, pages 41-42: Questions
Suppose that \( f(z) \) is a function which satisfies these two conditions. Then it follows that
\[
f(z) = f(x + iy) = f(x)f(iy) = e^x f(iy),
\]
so we need to determine what conditions will be imposed on the purely imaginary part of a complex number. Therefore, suppose that \( f(iy) = A(y) + iB(y) \), so we have \( f(x + iy) = e^x A(y) ie^x B(y) \). In order for \( f \) to be analytic, it needs to satisfy the Cauchy Riemann equations, so we must have \( u_x = v_y \) and \( u_y = -v_x \) or
\[
e^x A(y) = e^x B'(y)
\]
and
\[
e^x A'(y) = e^x B(y).
\]
This means that

\[ A''(y) = -A(y). \]

A general solution to this differential equation is

\[ A(y) = \alpha \cos (y) + \beta \sin (y) \]

for some real numbers \( \alpha \) and \( \beta \) and

\[ B(y) = -A'(y) = -\beta \cos (y) + \alpha \sin (y), \]

so we get

\[ f(z) = e^x(\alpha \cos (y) + \beta \sin (y)) + ie^x(-\beta \cos (y) + \alpha \sin (y)) \]

\[ = \alpha e^x(\cos (y) + i \sin (y)) + \beta e^x(\sin (y) - i \cos (y)). \]

If \( f(z) \) agrees with \( e^x \) for real numbers, we must have

\[ 1 = f(0) = \alpha - i\beta, \]

so \( \alpha = 1 \) and \( \beta = 0 \). Putting all this together, we have the following:

**Definition 2.1.** We define the complex exponential function \( f(z) = e^z \) as

\[ f(z) = f(x + iy) = e^x(\cos (y) + i \sin (y)). \]

**Proposition 2.2.** The exponential function satisfies the following:

(i) \[ |e^z| = e^x \]

(ii) \( e^z \) is not 0 for any value of \( z \)

(iii) \( e^{iy} = \text{cis}(y) \)

(iv) \( e^z = \alpha \) has infinitely many solutions for \( \alpha \neq 0 \).

(v) \( (e^z)' = e^z \)

**Proof.** Most of these results are fairly trivial to prove.

(i) \[ |e^z| = e^x |\cos (y) + i \sin (y)| = e^x (\cos^2 (y) + \sin^2 (y)) = e^x. \]

(ii) This simply follows because \( e^x \) is not 0 for any \( x \) and \( \cos (y) + i \sin (y) \) is not 0 for any \( y \).

(iii) \[ e^z = e^{x+iy} = xe^{iy} = e^x(\cos (y) + i \sin (y)) \]

so \( e^{iy} = \cos (y) + i \sin (y) \).

(iv) Suppose \( \alpha \) is some non-zero complex number. Then in polar form, we have \( \alpha = re^{i\theta} = rcis(\theta) \) for some \( r > 0 \) and some angle \( 0 \leq \theta < 2\pi \). It follows that solutions to \( e^z = e^x e^{iy} = \alpha \)

will be \( x = \ln (r) \) and \( y = \theta + 2k\pi \) for any integer \( k \).

(v) Recall that \( f'(z) = f_x(z) \), so \( (e^z)' = (e^x e^{iy})' = e^x e^{iy} = e^z \).
2.2. **Defining the Trigonometric Functions.** Now we have defined the exponential function, we can define trigonometric functions. First we observe that for real values of $y$, we have

\[ e^{iy} = \cos (y) + i \sin (y) \]

and

\[ e^{-iy} = \cos (y) - i \sin (y). \]

Thus for real values of $y$, we can define

\[ \sin (y) = \frac{1}{2i} (e^{iy} - e^{-iy}) \]

and

\[ \cos (y) = \frac{1}{2} (e^{iy} + e^{-iy}). \]

To define complex trigonometric, we simply extend these definitions to complex variables. Specifically, we define them as follows:

**Definition 2.3.** We define the complex functions $\sin$ and $\cos$ as

\[ \sin (z) = \frac{1}{2i} (e^{iz} - e^{-iz}) \]

and

\[ \cos (z) = \frac{1}{2} (e^{iz} + e^{-iz}). \]

The complex trig functions share many of the identities with their real counterparts. We illustrate with an example.

**Example 2.4.** We shall show that the identity $\cos^2 (z) + \sin^2 (z) = 1$ holds for complex numbers.

Using the definitions, we have

\[ \cos^2 (z) + \sin^2 (z) = \left[ \frac{1}{2} (e^{iz} + e^{-iz}) \right]^2 + \left[ \frac{1}{2i} (e^{iz} - e^{-iz}) \right]^2 \]

\[ = \frac{1}{4} (e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4} (e^{2iz} - 2 + e^{-2iz}) = 1 \]

Of course, they also have many differences (most notably, complex trig functions are not bounded).

**Homework:** (Pages 41 & 42) 2,4,6,8,11,13,14,16,17,19,21