Line Integrals and Entire Functions

"Defining an Integral for Complex Valued Functions"

In the following sections, our main goal is to show that every entire function can be represented as an everywhere convergent power series in z. In order to do this, we shall need to develop the concept of a line integral. In vector calculus, we introduced the idea of a line integral. The concept was motivated by the fact that the domain of a function of two variables is the whole plane, so when we integrate functions of two variables, we can integrate over any curve in the plane. Since the domain of a complex function is also a plane (the complex plane), in order to define the integral for a function of a complex variable, we shall have to use a similar idea.

1. Definition and Properties of the Line Integral

1.1. **The Definition of a Line Integral.** Before we define a line integral and consider how to calculate it, we need some preliminary definitions and results. First we consider the more straight forward case where the real and imaginary parts of a function of a complex variables both depend on some fixed variable t.

Definition 1.1. Let $f(t) = u(t) + iv(t)$ be any continuous complex-valued function of the real variable t with $a \leqslant t \leqslant b$. Then we define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Calculation of integrals such as these are straight forward single variable calculations. In order to define a general line integral, we need a way to represent curves in the line. As with vector calculus, this can be done through the use of parameterization. The following are important definitions regarding parameterized curves.

Definition 1.2. (*i*) Let $z(t) = x(t) + iy(t)$ $a \leqslant t \leqslant b$. The curve determined by $z(t)$ is called piecewise differentiable and we set

$$\dot{z}(t) = x'(t) + iy'(t)$$

if x and y are continuous on $[a, b]$ and continuously differentiable (C^1) on each subinterval $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$ of some partition of $[a, b]$.

(*ii*) The curve is said to be smooth if in addition $\dot{z}(t) \neq 0$ except at a finite number of points.

For the rest of the course, unless otherwise stated, all curves will be assumed to be smooth. We can now define a line integral.

Definition 1.3. Let C be a curve parameterized by $z(t)$, $a \leqslant t \leqslant b$ and suppose f is continuous at all points of $z(t)$. Then the integral of f along C is

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt.$$

Note that this is now a single variable complex valued function of t so can be calculated in the standard way.

1.2. **Independence of Path Parameterization.** The direction in which we travel along C will change the value of the integral, but it seems that how we travel along the curve C (once a direction is specified) should not affect the values of the integral i.e. if $z(t)$ and $w(t)$ are two different parameterizations of C, then the integral along C should be the same regardless of the parameterization we use. Though this can sometimes fail, under certain additional conditions imposed on the parameterizations, we can guarantee that this will always be the case.

Definition 1.4. The two curves C_1 parameterized by $z(t)$ with $a \leqslant t \leqslant b$ and C_2 parameterized by $w(t)$ with $c \leqslant t \leqslant d$ are said to be smoothly equivalent if there exists a $1-1$ C^1 mapping $\lambda\colon [c,d] \to [a,b]$ such that $\lambda'(t) \geqslant 0$ for all t and $w(t) = z(\lambda(t))$.

Proposition 1.5. *If C_1 and C_2 are smoothly equivalent, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Proof. Suppose that $f(z) = u(z) + iv(z)$, C_1 and C_2 are parameterized by $z(t) = x(t) + iy(t)$ and $w(t)$ respectively, and $\lambda\colon [c,d] \to [a,b]$ is a $1-1$ C^1 the mapping with $w(t) = z(\lambda(t)) = x(\lambda(t)) + iy(\lambda(t))$ and $\lambda'(t) \geqslant 0$. Then we have

$$\int_{C_1} f(z)dz = \int_a^b f(z(t))\dot{z(t)}dt$$

$$= \int_a^b u(z(t))x'(t)dt - \int_a^b v(z(t))y'(t)dt + i\int_a^b u(z(t))y'(t)dt + \int_a^b v(z(t))x'(t)dt$$

and

$$\int_{C_2} f(z)dz = \int_c^d f(w(t))\dot{w(t)}dt = \int_c^d f(z(\lambda(t)))\dot{z(\lambda(t))}dt$$

$$= \int_c^d [u(z(\lambda(t))) + iv(z(\lambda(t)))][x'(\lambda(t)) + iy'(\lambda(t))]\lambda'(t)dt$$

$$= \int_c^d [u(z(\lambda(t)))x'(\lambda(t))\lambda'(t)]dt - \int_c^d [v(z(\lambda(t)))y'(\lambda(t))\lambda'(t)]dt$$

$$+i\int_c^d [u(z(\lambda(t)))y'(\lambda(t))\lambda'(t)]dt + \int_c^d [v(z(\lambda(t)))y'(\lambda(t))\lambda'(t)]dt$$

Comparing each of the four integrals, we see it is a direct consequence of the change of variable formulas for real integrals (i.e. substitution) that these are equal. For example, if we take $s = \lambda(t)$ in the integral

$$\int_c^d [u(z(\lambda(t)))x'(\lambda(t))\lambda'(t)]dt$$

we have $ds = \lambda'(t)dt$ and when $t = c$, $s = a$ and when $t = d$, $s = b$, so we get

$$\int_c^d [u(z(\lambda(t)))x'(\lambda(t))\lambda'(t)]dt = \int_a^b u(z(s))x'(s)ds$$

□

This shows that a line integral is independent of parameterization. However, as we stated before, it does depend upon direction. In order to illustrate this fact, we need the following definition.

Definition 1.6. Suppose C is given by $z(t)$ with $a \leqslant t \leqslant b$. Then $-C$ is defined by $z(b + a - t)$ with $a \leqslant t \leqslant b$ (so $-C$ is C traced in the opposite direction.

Proposition 1.7.

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

Proof. We have

$$\int_{-C} f(z)dz = \int_a^b f(z(b + a - z))\dot{z}'(b + a - t)dt$$

Similar to the last question, we can distribute to obtain four different real integrals and then make the substitution $s = b + a - z$ to each of the real functions to prove the equality.

□

We consider some examples.

Example 1.8. (i) Calculate

$$\int_C \frac{1}{z}dz$$

where $z(t) = \cos{(t)} + i\sin{(t)}$ with $0 \leqslant t \leqslant 2\pi$.

First observe that

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}.$$

Then we have

$$\int_C \frac{1}{z}dz = \int_0^{2\pi} (\cos(t) - i\sin(t))(-\sin(t) + i\cos(t))dt = \int_0^{2\pi} idt = 2\pi i$$

(ii) Calculate

$$\int_C \frac{1}{z}dz$$

where C is the square of side length 1 oriented counterclockwise centered at the origin.

We need to calculate the integral over all four sides. Since all calculations are similar, we calculate only over C_1, the top side parameterized by $z(t) = -t + i$ with $-1 \leqslant t \leqslant 1$. Here we have

$$\int_{C_1} \frac{1}{z}dz = \int_{-1}^{1} \left(-\frac{t}{t^2+1} + \frac{i}{t^2+1} \right)(1)dt$$

$$= -\frac{\ln(t^2+1)}{2}\bigg|_{-1}^{1} + i\arctan(t)\bigg|_{-1}^{1} = \frac{\pi i}{2}.$$

It can be shown that the integral over each side is also $i\pi/2$, so we get

$$\int_C \frac{1}{z}dz = 2\pi i$$

(observe that this is the same as the previous question - we shall see why later).

1.3. **Preliminary Results for Line Integrals.** We now consider generalizing some results for line integrals of real variables to line integrals of complex variables. The first result shows that the operation of integration is linear and can be proved by considering the corresponding real integrals and using the real result.

Proposition 1.9. *Let C be a smooth curve, let f and g be continuous functions of z and let α be any complex number. Then*

(i)

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

(ii)

$$\int \alpha f(z)dz = \alpha \int_C f(z)dz$$

The next result we consider generalizes the idea that the absolute value of a definite integral of a function is bounded by the integral of the absolute value of the same functions.

Lemma 1.10. *Suppose $G(t)$ is a continuous and complex valued function of t. Then*

$$\left| \int_a^b G(t)dt \right| \leqslant \int_a^b |G(t)|dt.$$

Proof. Suppose that

$$\int_a^b G(t)dt = Re^{i\vartheta}$$

for some fixed ϑ and $R \geqslant 0$. Then observe that

$$\left| \int_a^b G(t)dt \right| = R$$

so we just need to show that

$$R \leqslant \int_a^b |G(t)|dt.$$

In order to do this, we consider the function $e^{-i\vartheta}G(t)$. First, using the linearity properties of the integral, we have

$$\int e^{-i\vartheta}G(t)dt = R$$

so if

$$e^{-i\vartheta}G(t) = A(t) + iB(t)$$

(as real and imaginary parts), also using linearity of the integral, we must have

$$R = \int_a^b A(t)dt.$$

But

$$A(t) = Re(e^{-i\vartheta}G(t)) \leqslant |Re(e^{-i\vartheta}G(t))| \leqslant |e^{-i\vartheta}G(t)| = |G(t)|$$

so we have

$$R = \int_a^b A(t)dt \leqslant \int_a^b |G(t)|dt$$

so the result follows.

□

This result now allows us to impose an upper bound on any line integral (similar to real variable where an upper bound is given by the length of the curve multiplied by the maximum value of the function on the curve).

Theorem 1.11. *$M - L$ Formula - Suppose that C is a smooth curve of length L on G and that $|f(z)| \leqslant M$ throughout C. Then*

$$\left| \int_C f(z)dz \right| \leqslant ML.$$

Proof. We have

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))\dot{z}(t)dt\right| \leqslant \int_a^b |f(z(t))||\dot{z}(t)|dt \leqslant \int_a^b M|\dot{z}(t)|dt$$

$$= M\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt = ML$$

since

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt = L$$

is the formula for the length of a curve parameterized by $z(t)$ (recall Vector Calculus and Calculus 2). $\square$

Example 1.12. Show that

$$\left|\int_C \frac{1}{z^2}dz\right| \leqslant 2\pi$$

on the unit circle (oriented in either direction).
For this we simply observe that $|1/z^2| \leqslant 1$ on the unit circle, so

$$\left|\int_C \frac{1}{z^2}dz\right| \leqslant 2\pi.$$

We can use he ML-formula to show that the integral of a sequence of uniformly integrals converge.

Proposition 1.13. *Suppose that $\{f_n\}$ is a sequence of functions and $f_n \to f$ uniformly on a smooth curve C. Then*

$$\int_C f(z)dz = \lim_{n\to\infty} \int_C f_n(z)dz.$$

Proof. First by linearity of the integral, we have

$$\int_C f(z)dz - \int_C f(z)dz = \int_C (f(z) - f_n(z))dz.$$

Since $f_n \to f$ uniformly, for n sufficiently large, we have $|f - f_n| < \varepsilon$ for any $\varepsilon > 0$. Then we have

$$\left|\int_C (f(z) - f_n(z))dz\right| \leqslant \varepsilon L$$

where L is the length of the curve. Since ε can be as small as we please, it follows that

$$\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz.$$

$\square$

The last preliminary result we prove is a generalization of the fundamental theorem of calculus for real variables.

Proposition 1.14. *Suppose that $F' = f$ where F is an analytic function which is smooth on C. Then*

$$\int_C f(z)dz = F(z(b)) - F(z(a)).$$

Proof. We shall prove this by showing that the derivative of $F(z(t))$ with respect to t is $f(z(t))\dot{z}(t)$ and then use the Fundamental Theorem from real variable calculus to finish.

First note that since F is analytic, $F(z(t))$ will be a smooth curve. It follows that

$$F(\dot{z}(t)) = \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{F(z(t+h)) - F(z(t))}{h}$$

i.e. since $F(z(t))$ is smooth, the derivative is defined so can be calculated using the difference quotient taking h in any direction toward 0.

Next we note that since $\dot{z}(t) \neq 0$, (except for a finite number of points), and is continuous, we can find δ such that $|h| < \delta$ implies $z(t+h) - z(t) \neq 0$. Thus we have

$$F(\dot{z}(t)) = \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{F(z(t+h)) - F(z(t))}{h}$$

$$= \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h}$$

$$= F'(z(t)) \cdot \dot{z}(t) = f(z(t))\dot{z}(t).$$

Then by definition and using the Fundamental Theorem of Calculus for real variables, we have

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt = F(z(b)) - F(z(a)),$$

hence the result.

□

2. The Closed Curve Theorem for Entire Functions

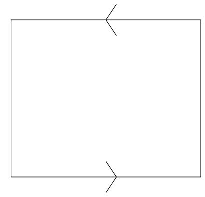
In this section we consider the generalization of the result in multivariable calculus which states that the line integral of a gradient vector over a closed curve is 0. In order to prove this result, we shall have to prove the result for certain special curves first.

2.1. **The Rectangle Theorem.**

We shall first consider integrals over rectangles. We need the following definitions.

Definition 2.1. A curve C is closed if its initial points and terminal points coincide.

Definition 2.2. By the boundary of a rectangle, we mean a simple closed curve parameterized so that the rectangle it traces is on the left ofs the curve is traced (see picture).



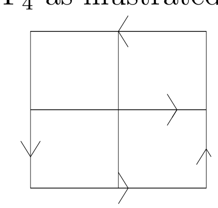
Theorem 2.3. *Suppose f is an entire function and Γ is the boundary of a rectangle R. Then*

$$\int_{\Gamma} f(z)dz = 0.$$

Proof. In order to prove this result, we first note that by the fundamental theorem of calculus, the result holds for linear functions i.e. $f(z) = az + b$ is everywhere the derivative of $az^2/2 + bz$ and the endpoints of the curve R are the same. Thus the result holds for linear functions.
Now suppose that $f(z)$ is an arbitrary entire function and let

$$I = \int_{\Gamma} f(z)dz.$$

We can break up the rectangle R into 4 equally sized rectangles with boundaries Γ_1, Γ_2, Γ_3 and Γ_4 as illustrated below.



Observe that the sum of the integrals around each of these rectangles is equal to the integral around R since the integrals over the sides inside R cancel each other out. Thus we have

$$\int_{\Gamma} f(z)dz = \sum_{i=1}^{4} \int_{\Gamma_i} f(z)dz.$$

It follows that for at least one of these integrals, which we shall denote by $\Gamma^{(1)}$, we have

$$\left| \int_{\Gamma^{(1)}} f(z)dz \right| \geqslant \frac{I}{4}.$$

Let R^1 denote the rectangle with boundary $\Gamma^{(1)}$.
We can do exactly the same thing with the rectangle $R^{(1)}$ and continue this process until we obtain a sequence of rectangles with

$$R^{(1)} \supset R^{(2)} \supset \ldots$$

with boundaries

$$\Gamma^{(1)} \supset \Gamma^{(2)} \supset \dots,$$

with the following properties:

(*i*)

$$\text{Side lengths} R^{(k+1)} = (\text{ Side lengths} R^{(k)})/2$$

(*ii*)

$$\left| \int_{\Gamma^{(k)}} f(z) dz \right| \geqslant \frac{I}{4^k}.$$

Now observe that since f is entire, at any $z_0 \in R$, there exists ε_z such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z - z_0)$$

where $\varepsilon_z \to 0$ as $z \to z_0$ i.e. it is differentiable, so can be approximated using local linearization. Thus we have

$$\int_{\Gamma^{(n)}} f(z)dz = \int_{\Gamma^{(n)}} f(z_0)+f'(z_0)(z-z_0)+\varepsilon(z-z_0)dz = \int_{\Gamma^{(n)}} \varepsilon_z(z-z_0)dz$$

since the rest is linear. Thus we just need to bound this integral.
Let s denote the length of the largest side of Γ. Then we have

$$\text{length of } \Gamma^{(n)} \leqslant \frac{4s}{2^n}$$

(since s has the largest length) and

$$|z - z_0| \leqslant \frac{\sqrt{2}s}{2^n}$$

for all $z \in \Gamma^{(n)}$ (the length from one corner of the square of side length $s/2^n$ to the diagonal opposite - the two furthest points) .
Next note that since $\varepsilon_z \to 0$ as $z \to z_0$, for any ε, we can find N such that

$$|z - z_0| \leqslant \frac{\sqrt{2}s}{2^N}$$

implies $\varepsilon_z < \varepsilon$. Then using the ML formula, for $n \geqslant N$, we have

$$\left| \int_{\Gamma^{(n)}} f(z)dz \right| \leqslant \varepsilon \frac{4s}{2^n} \cdot \frac{\sqrt{2}s}{2^n} = \frac{4\sqrt{2}s^2}{4^n}\varepsilon.$$

It follows that

$$\frac{|I|}{4^n} \leqslant \frac{4\sqrt{2}s^2}{4^n}\varepsilon$$

or equivalently

$$|I| \leqslant 4\sqrt{2}s^2\varepsilon.$$

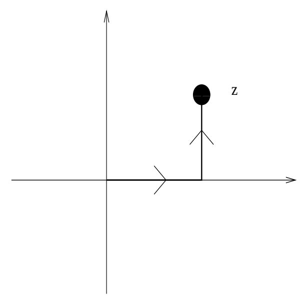
This is now independent of z and holds for all epsilon, so it follows that $I = 0$.

$\square$

2.2. **The Integral Theorem.** Our next task is to show that any entire function $f(z)$ is the derivative of some analytic function $F(z)$ which we will then be able to use to prove that the integral of an entire function around any closed curve is 0.

Theorem 2.4. *(The Integral Theorem) If $f(z)$ is entire, then f is everywhere the derivative of some analytic function i.e. there exists $F(z)$ such that $f(z) = F'(z)$ for all z.*

Proof. We define $F(z) = \int_0^z f(\zeta)d\zeta$ where $\int_0^z$ denotes the integral along the straight lines from 0 to $Re(z)$ and from $Re(z)$ to $Im(z)$ (see illustration below).



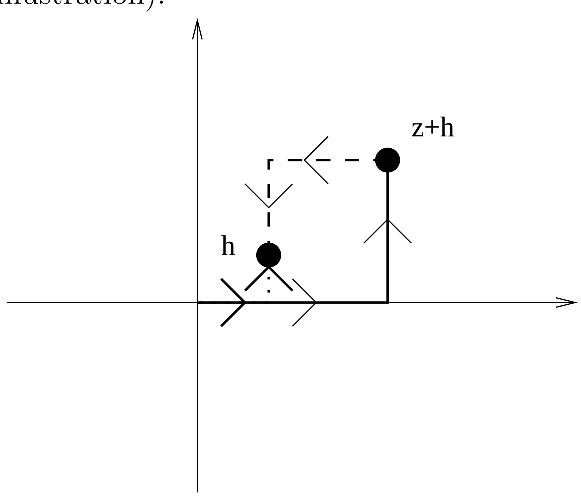
Next note that

$$F(z+h) - F(z) = \int_0^h f(\zeta)d\zeta.$$

We can see this by looking at the lines over which the integrals are being calculated. Specifically,

$$F(z+h) - F(z) - \int_0^h f(\zeta)d\zeta$$

is the integral of the function $f(\zeta)$ from $(0,0)$ to $Re(z)$, followed by the rectangle with corners $Re(h)$, $Re(z+h)$, $z+h$ and $h+Im(z)$ in a counterclockwise direction and finishing with the integral from $Re(z)$ to $(0,0)$ (see illustration).



The rectangle theorem tells us the integral over the rectangle will be zero, and the other two integrals cancel (since they are along the same line in opposite directions) so

$$F(z+h) - F(z) - \int_0^h f(\zeta)d\zeta = 0.$$

Using this equality, we next observe that

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_0^h [f(\zeta) - f(z)]d\zeta$$

since the integral does not depend upon z, so

$$\int_0^h f(z)dz = f(z)h.$$

Since we are trying to show that $F'(z) = f(z)$, we need to show that

$$\frac{F(z+h) - F(z)}{h} - f(z) \to 0$$

so our observations imply that it suffices to show that

$$\frac{1}{h}\int_0^h [f(\zeta) - f(z)]d\zeta \to 0.$$

Since f is continuous, for sufficiently small h we can guarantee that $|f(z) - f(\zeta)| < \varepsilon$ for any ζ, so we get

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_0^h [f(\zeta) - f(z)]d\zeta\right| \leqslant \frac{1}{|h|}\varepsilon|h| = \varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \to 0$$

so $F'(z) = f(z)$.

□

2.3. **The Closed Curve Theorem.** We are now ready to prove the main result of this section - that the line integral of an entire function around any closed curve is 0. With the results we already have, the proof is now straight forward.

Theorem 2.5. *If f is entire and C is a smooth closed curve, then*

$$\int_C f(z)dz = 0.$$

Proof. Since f is entire, we have $f(z) = F'(z)$ for some analytic function $F(z)$. Then

$$\int_C f(z)dz = \int_C F'(z)dz = F(z(b)) - F(z(a)) = 0$$

since $F(z(b)) = F(z(a))$.

$\square$

In this Theorem we stated that f was entire, but this theorem can be generalized to any region on which f is the derivative of some analytic function $F(z)$. Specifically, the more general closed curve theorem would be the following:

Theorem 2.6. *If f is entire on some region D (which is not necessarily the whole complex plane) containing the smooth closed curve C, then*

$$\int_C f(z)dz = 0.$$

Note that this theorem fails if the function is not entire on a region that contains C as we saw when we showed

$$\int_C \frac{1}{z}dz = 2\pi i$$

for the unit circle C centered at the origin.

Homework:
Questions from pages 54-55; 2,3,4,5,7,9