Properties of Entire Functions

"Generalizing Results to Entire Functions"

Our main goal is still to show that every entire function can be represented as an everywhere convergent power series in z. So far we have developed the notion of a line integral and this will be one of the integral parts of this goal. In this chapter, we generalize some of the results of entire functions to other related functions and then use it to prove the main result. Following this, we shall consider a couple of interesting consequences.

1. The Cauchy Integral Formula and Taylor Expansion for Entire Functions

1.1. **The Rectangle Theorem Revisited.** Before we prove that every entire function can be represented by a power series, we need a few preliminary results. The first result is a proof of the rectangle theorem for a function closely related to some given entire function $f(z)$.

Theorem 1.1. *If f is entire and*

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then

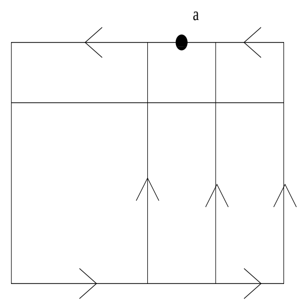
$$\int_\Gamma g(z)dz = 0$$

where Γ is the boundary of any rectangle R.

Proof. The proof breaks into three cases depending upon whether a is outside the rectangle, inside the rectangle or on the rectangle.

I. a lies on the outside of the rectangle.
In this case the function $g(z)$ is analytic inside and on R, so the proof is identical to that given in the original rectangle theorem (where all we needed was that it was analytic inside and on the rectangle).

II. a lies on the rectangle.
In this case we subdivide the rectangle up into smaller rectangles with sides Γ_i with $1 \leqslant i \leqslant 6$ as follows.



Due to cancellation along most of the interior sides, we see that

$$\int_\Gamma g(z)dz = \sum_{i=1}^{6} \int_{\Gamma_i} g(z)dz.$$

Let R_1 be the rectangle with a in one of its sides and Γ_1 its sides. As with the last case, we can use the rectangle theorem to show

$$\int_{\Gamma_i} g(z)dz = 0$$

for $2 \leqslant i \leqslant 6$, so it follows that

$$\int_\Gamma g(z) = \int_{\Gamma_1} g(z)dz.$$

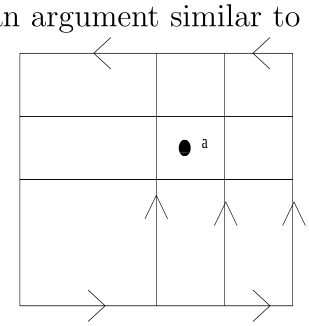
Thus we just need to calculate this integral. Now since $g(z)$ is continuous and R is closed and bounded, it must take a maximum value somewhere in R i.e. $|g(z)| \leqslant M$ for all $z \in R$ some fixed M. If we choose the rectangle R to have total sidelengths ε, using the ML formula, we have

$$|\int_{\Gamma_1} g(z)|dz \leqslant M\varepsilon.$$

Since this works for any ε, it follows that

$$|\int_\Gamma g(z)|dz = 0.$$

III. a lies on the interior of the rectangle.
In this case we break up the rectangle into nine individual rectangles as illustrated and use an argument similar to above.



In particular, the integrals over each of the rectangles which do not contain a will be zero, the value of $g(z)$ will always be bounded, so by choosing the rectangle containing a to be sufficiently small, we can make that integral as small as we like, so it follows that the integral over Γ is 0.

□

An immediate consequence of the rectangle theorem is the following.

Corollary 1.2. *If f is entire and*

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then the integral theorem ($g(z)$ is the derivative of some entire function) and the closed curve theorem (the integral around any closed curve is 0*0 apply to $g(z)$.*

Proof. Both of these results were consequences of the rectangle theorem, so the proofs are identical for $g(z)$. □

1.2. **The Cauchy Integral Formula and Consequences.** Now we have shown that the rectangle theorem and other results hold for

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

we are ready to prove one of the most important results in complex analysis which provides a way to integrate functions which are not entire.

Theorem 1.3. *(The Cauchy Integral Formula) Suppose that f is entire, a is some complex number and C is the curve $C\colon Re^{i\vartheta}$, $0 \leqslant \vartheta \leqslant 2\pi$ with $R > |a|$ (so a is in the interior of C). Then*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz.$$

Proof. Since C is closed, by the previous result, we know

$$\int_C \frac{f(z)-f(a)}{z-a}dz = 0$$

so it follows that

$$f(a)\int_C \frac{1}{z-a}dz = \int_C \frac{f(z)}{z-a}dz.$$

Thus it suffices to show that

$$\int_C \frac{1}{z-a}dz = 2\pi i$$

which we do in the following Lemma.

□

Lemma 1.4. *Suppose a is contained the circle C_ϱ with center α and radius ϱ. Then*

$$\int_{C_\varrho} \frac{dz}{z-a} = 2\pi i.$$

Proof. First note that C_ϱ is parameterized by $z(t) = \varrho e^{it} + a$ for $0 \leqslant t \leqslant 2\pi$. It follows that

$$\int_{C_\varrho} \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{i\varrho e^{it}}{\varrho e^{it}} dt = 2\pi i.$$

Also observe that for $k > 1$, we have

$$\int_{C_\varrho} \frac{dz}{(z-\alpha)^{k+1}} = \int_0^{2\pi} \frac{i\varrho e^{it}}{\varrho^k e^{(k+1)it}} dt = \frac{i}{\varrho^k} \int_0^{2\pi} e^{-itk} dt = -\frac{i}{\varrho^k} \frac{e^{-itk}}{ik}\bigg|_0^{2\pi} = 0.$$

To evaluate

$$\int_{C_\varrho} \frac{dz}{z-a}$$

we write

$$\frac{1}{z-a} = \frac{1}{(z-\alpha)-(a-\alpha)} = \frac{1}{(z-\alpha)[1-(a-\alpha)/(z-\alpha)]}$$

$$= \frac{1}{z-\alpha} \cdot \frac{1}{1-(a-\alpha)/(z-\alpha)}$$

Since $|(a-\alpha)/(z-\alpha)| = |(a-\alpha)|/|\varrho| < 1$ on the circle C_ϱ,

$$\frac{1}{1-(a-\alpha)/(z-\alpha)}$$

can be expressed as a geometric series which converges uniformly to

$$\frac{1}{1-(a-\alpha)/(z-\alpha)}.$$

Specifically, on C_ϱ, we have

$$\frac{1}{1-(a-\alpha)/(z-\alpha)} = 1 + \frac{a-\alpha}{z-\alpha} + \left(\frac{a-\alpha}{z-\alpha}\right)^2 + \dots.$$

Thus we have

$$\int_{C_\varrho} \frac{dz}{z-a} = \int_{C_\varrho} \frac{1}{z-\alpha} \left[\sum_{k=0}^{\infty} \left(\frac{a-\alpha}{z-\alpha}\right)^k \right] dz = \sum_{k=0}^{\infty} (a-\alpha)^k \int_{C_\varrho} \frac{dz}{(z-\alpha)^{k+1}}$$

$$= \int_{C_\varrho} \frac{dz}{z-\alpha} = 2\pi i$$

thus completing the proof. $\square$

1.3. **The Power Series Representation of an Entire Function.** Now we have Cauchy's Theorem, we are now able to prove the major first result we shall see in complex analysis - the fact that any entire complex function can be represented as a power series.

Theorem 1.5. *(Taylor Expansion of an Entire Function) Suppose that $f(z)$ is an entire function. Then it has a power series representation, and in fact $f^k(0)$ exists for every k and*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} z^k.$$

Proof. In Chapter 2, we showed that if a power series

$$\sum a_k z^k$$

has a nonzero radius of convergence, then the coefficients of the power series are determined by its value at 0 i.e. if $f(z) = \sum a_k z^k$ has a nonzero radius of convergence, then $a_k = f^k(0)/k!$. Therefore, if we can show show that $f(z)$ can be represented as some power series at every point of the complex plane, the result will follow.
Suppose $a \neq 0$, $R = |a| + 1$ and let C be the circle centered at the origin with radius R. Then for any z in the circle C, the Cauchy integral formula implies

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{z - \omega} d\omega.$$

Observe that

$$\frac{1}{z-\omega} = \frac{1}{\omega(1 - \frac{z}{\omega})}$$

and since z lies within the circle, we have $|z/w| < 1$. In particular, we can represent

$$\frac{1}{\omega(1 - \frac{z}{\omega})}$$

as a uniformly convergent power series in C i.e

$$\frac{1}{\omega(1 - \frac{z}{\omega})} = \frac{1}{\omega} + \frac{z}{\omega^2} + \frac{z^2}{\omega^3} + \dots$$

so we get

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{z - \omega} d\omega = \frac{1}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{f(\omega) z^k}{\omega^{k+1}} d\omega$$

$$= \sum_{k=0}^{\infty} z^k \left(\frac{1}{2\pi i} \int_C \frac{f(\omega)}{w^{k+1}} d\omega \right) = \sum_{k=0}^{\infty} a_k z^k$$

where

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{w^{k+1}} d\omega.$$

Thus for any z in C, $f(z)$ can be represented as a power series. This result holds for any circle centered at the origin and hence for any z in the complex plane, $f(z)$ can be represented as a convergent series.

□

The following results are consequences of this important Theorem.

Corollary 1.6. *An entire function is infinitely differentiable.*

Proof. This follows from our results on power series and the fact that $f(z)$ can be represented as a power series on the whole complex plane.

□

Corollary 1.7. *If f is entire ad a is any complex number, then*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots$$

for all z.

Proof. We just make the shift $z \to z - a$ and apply the results we have already developed. □

Proposition 1.8. *If f is entire and g is defined by*

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then $g(z)$ is entire.

Proof. By the previous result, since f is entire, for $\neq a$, we have

$$g(z) = \frac{f(z)-f(a)}{z-a} = f'(a) + \frac{f''(a)}{2!}(x-a) + \frac{f^3(a)}{3!}(x-a)^2 + \dots.$$

By the definition of $g(z)$, the value $f'(a)$ agrees with this power series at $z = a$. Thus $g(z)$ is representable as an everywhere convergent power series and hence the results of Chapter 2 imply g is differentiable everywhere and so entire. □

Corollary 1.9. *If f is entire with zeros at $a_1, \dots a_N$. Then if g is defined by*

$$g(z) = \frac{f(z)}{(z-a_1)(z-a_2)\dots(z-a_N)}$$

for $z \neq a_k$ then $\lim_{z \to a_k} g(z)$ exists for $k = 1, \dots N$ and if g is defined by these limits, then g is entire.

Proof. Let $f_0(z) = f(z)$ and define

$$f_k(z) = \frac{f_{k-1}(z) - f_{k-1}(a_k)}{z - a_k} = \frac{f_{k-1}(z)}{z - a_k}$$

for $z \neq a_k$. Assuming $f_{k-1}(z)$ is entire, it follows from the previous result that the function

$$h(z) = \begin{cases} \frac{f_{k-1}(z) - f_{k-1}(a_k)}{z - a_k} & z \neq a_k \\ f'_{k-1}(a) & z = a_k \end{cases}$$

is an entire function, so in particular, the limit of $f_k(z)$ must exist at $z = a_k$. Thus if we define $f_k(a_k) = f'_{k-1}(a_k)$, then $f_k(z)$ will be entire. Since $f(z)$ is entire, the result follows by induction since $g(z) = f_N(z)$.

□

2. Applications - Liouville's Theorems and the Fundamental Theorem of Algebra

We finish by considering some applications of the results we have proved for entire functions.

Theorem 2.1. *(Liouville's Theorem) A bounded entire function is constant.*

Proof. For any $a, b \in \mathbb{C}$ we have

$$|f(b) - f(a)| = \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{z-b} dz \right|$$
$$= \left| \frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} \right|$$

where C is a circle centered at the origin whose interior contains a and b. Since we are assuming that $f(z)$ is bounded, it follows that $|f(z)| \leqslant M$ for all z. Also note that if R is the radius of C, then

$$\left| \frac{1}{(z-a)(z-b)} \right| \leqslant \frac{1}{(R - |a|)(R - |b|)}$$

for any z on C since $R - |a| = |z| - |a| \leqslant |z - a|$ (and similarly with b). Then since the length of C is $2\pi R$, using ther ML-formula, we get

$$\left| \frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} dz \right| \leqslant \frac{1}{2\pi} \frac{2\pi M|b-a|}{(R-|a|)(R-|b|)} = \frac{M|b-a|}{(R-|a|)(R-|b|)}.$$

Since R can be taken as large as we like, and R, a and b are all fixed, taking $R \to \infty$ we see

$$\left| \frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} dz \right| \leqslant \frac{M|b-a|}{(R-|a|)(R-|b|)} \to 0,$$

hence the result.

□

Theorem 2.2. *(The Extended Liouville Theorem) If $f(z)$ is entire and if for some integer k there exists positive constants A and B such that*

$$|f(z)| \leqslant A + B|z|^k$$

then f is a polynomial of degree at most k.

Proof. We prove this result by induction. First note that $k = 0$ is the original Liouville Theorem. Now suppose the result holds for a given k and assume that $f(z)$ is an entire function such that $|f(z)| \leqslant A + B|z|^{k+1}$ for some positive constants A and B. Define $g(z)$ as

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}.$$

Observe that since

$$|f(z)| \leqslant A + B|z|^{k+1},$$

it follows that

$$|g(z)| \leqslant C + D|z|^{k}$$

for some C and D (by the way that $g(z)$ is defined). Since g is entire, the induction hypothesis implies that $g(z)$ if a polynomial of degree at most k, and thus it follows that $f(z)$ is also a polynomial which has degree at most $k + 1$.

□

One useful application of Liouville's Theorem is a proof of the fundamental theorem of algebra - that is a proof of the fact that every non constant polynomial with complex coefficients has a zero in the complex numbers.

Theorem 2.3. *(The Fundamental Theorem of Algebra) Every non contsant polynomial with complex coefficients has a zero in $\mathbb{C}$.*

Proof. Let $P(z)$ be a polynomial and suppose $P(z) \neq 0$ for any $z \in \mathbb{C}$. It follows that $f(z) = 1/P(z)$ is an entire function. Moreover, if P is non-constant then $P \to \infty$ as $z \to \infty$, so $f(z)$ is bounded. Applying Liouville's Theorem, it follows that $f(z)$ is constant, hence so is $P(z)$ contrary to our assumption.

□

The fundamental theorem of algebra guarantees the existence of non constant polynomials, and through the use of induction, it can be shown that a degree n polynomial must in fact have n zeros, though some of them may occur more than once. With this in mind, we define the following.

Definition 2.4. For a polynomial $p(z)$, $\alpha \in \mathbb{C}$ is called a zero of multiplicity k if $P(z) = (z - \alpha)^k Q(z)$, but α is not a zero of $Q(z)$.

Homework:
Questions from pages 54-66; 2,3,4,5,8,10,12,14