Properties of Analytic Functions

"Generalizing Results to Analytic Functions"

In the last few sections, we completely described entire functions through the use of everywhere convergent power series. Our goal for the following sections is to instead consider functions which are analytic almost everywhere and see which results generalize. We shall see that we do get some generalizations (like representations of functions via convergent power series), but unlike the entire case, we need to impose domain restrictions on the corresponding power series.

1. The Power Representation of a Function which is Analytic in a disc.

Our first task is to show that if a function is analytic in a disc, then it can be represented by a power series which is convergent in that disc. The results and proofs we use shall be similar to many we have already developed so we shall skip many of the steps.

Theorem 1.1. *(Another Rectangle Theorem) Suppose that f is analytic in a disc $D(\alpha, r)$. If the closed rectangle R and the point a are both contained in D and Γ represents the boundary of R, then*

$$\int_\Gamma f(z)dz = \int_\Gamma \frac{f(z) - f(a)}{z - a} dz.$$

Proof. This is identical to the proof for entire functions since we only used the fact that it was analytic inside and on the rectangle bounded by Γ.

□

For ease of notation, for a fixed $f(z)$ which is analytic in a circle D containing the point $z = a$, we denote by $g(z)$ the function defined by

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \in D, z \neq a \\ f'(a) & z = a \end{cases}$$

Our next task is to show that $f(z)$ and $g(z)$ are the derivatives of functions defined in the disc (which will be a generalization of the same result for entire functions proved using the rectangle theorem).

Theorem 1.2. *If f is analytic in the disc $D(\alpha, r)$ and $a \in D(\alpha, r)$, there exists functions F and G such that*

$$F'(z) = f(z)$$

and

$$G'(z) = g(z).$$

Proof. We define

$$F(z) = \int_{\alpha}^{z} f(\zeta)d\zeta$$

and

$$G(z) = \int_{\alpha}^{z} g(\zeta d\zeta$$

where the path of integration goes along horizontal and vertical line segments in D. Next observe that for any $z \in D$ and h small enough, we can guarantee that $z+h \in D$, so we can apply the rectangle theorem as we did for the entire case to the difference quotients to obtain the result.

□

Theorem 1.3. *For f, g and a as before, if C is any closed curve in D, then*

$$\int_C f(z)dz = \int_C g(z)dz = 0.$$

Proof. By the last result, we know there exists F with $F' = f$ in D, so if C is parameterized by $z(t)$ with $a \leqslant t \leqslant b$, we have

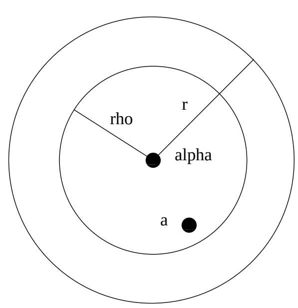
$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z(t)}dt = F(z(b)) - F(z(a)) = 0$$

since the initial and endpoints are the same. An identical result holds for $g(z)$.

□

Theorem 1.4. *(The Cauchy Integral Formula) Suppose f is analytic in $D(\alpha, r)$, $0 < \varrho < r$ and $|a - \alpha| < \varrho$ (so there is a circle which we denote by C_ϱ of radius ϱ centered on α which contains the point a and is fully contained in D (so C is parameterized by $\alpha + \varrho e^{i\vartheta}$ for $0 \leqslant \vartheta < 2\pi$ - see illustration). Then*

$$f(a) = \frac{1}{2\pi i}\int_{C_\varrho} \frac{f(z)}{z-a}dz.$$



Proof. Since

$$\int_{\mathbb{C}_\varrho} \frac{f(z) - f(a)}{z - a} dz = 0$$

we have

$$f(a) \int_{C_\varrho} \frac{dz}{z - a} = \int_{C_\varrho} \frac{f(z)}{z - a} dz.$$

However, we saw earlier that

$$\int_{C_\varrho} \frac{dz}{z - a} = 2\pi i$$

so we are done.

□

With all these results from entire functions generalized to functions which are analytic in a disc, we are now ready to prove the main result for analytic functions in a disc - the fact that they can be represented by a power series which is convergent in that disc.

Theorem 1.5. *If f is analytic in a disc $D(\alpha, r)$, then there exists constants C_k such that*

$$f(z) = \sum_{k=1}^{\infty} C_k (z - \alpha)^k$$

for all $z \in D$.

Proof. We follow a similar proof to the one given for an entire function. Suppose $a \in D$ and choose ϱ such that $|a - \alpha| < \varrho < r$. By the previous result, for any z with $|z - \alpha| < |a - \alpha|$ we have

$$f(z) = \frac{1}{2\pi i} \int_{C_\varrho} \frac{f(w)}{w - z} dw.$$

Next using the fact that

$$\frac{1}{w - z} = \frac{1}{w - \alpha} + \frac{z - \alpha}{(w - \alpha)^2} + \frac{(z - \alpha)^2}{(w - \alpha)^3} + \dots$$

and the series converges uniformly throughout C_ϱ, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_\varrho} f(w) \left(\frac{1}{w - \alpha} + \frac{z - \alpha}{(w - \alpha)^2} + \frac{(z - \alpha)^2}{(w - \alpha)^3} + \dots \right) dw \\ &= \sum_{k=1}^{\infty} (z - \alpha)^k \frac{1}{2\pi i} \int_{C_\varrho} \frac{f(w)}{(w - \alpha)^{k+1}} dw \end{aligned}$$

i.e. $f(z)$ can be represented as the power series

$$\sum_{k=0}^{\infty} C_k (z - \alpha)^k$$

where
$$C_k = \frac{1}{2\pi i}\int_{C_\varrho}\frac{f(w)}{(w-\alpha)^{k+1}}dw.$$
As before, we note that these bounds appear to depend upon ϱ, but using the uniqueness theorem for power series in their domain of convergence, we see that this in not the case. Thus for all $z \in D$,
$$f(z) = \sum_{k=0}^{\infty} C_k(z-\alpha)^k$$
where
$$C_k = \frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i}\int_{C_\varrho}\frac{f(w)}{(w-\alpha)^{k+1}}dw.$$
□

It should be pointed out that the results we have proved **do not** generalize to arbitrary open sets even though it seems like it should - there is something very special about convergence of power series and discs of convergence. The following result however can be derived in a similar way to the results we have just proved.

Theorem 1.6. *If f is analytic in an arbitrary open domain D, then for each $\alpha \in D$, there exists constants C_k such that*
$$f(z) = \sum_{k=0}^{\infty} C_k(z-\alpha)^k$$
for all points z inside the largest disc centered at α and contained in D.

Example 1.7. (i) Determine the power series representation for
$$f(z) = \frac{1}{(z-1)}$$
for $|z| < 1$.

Observe that around $z = 0$, we have
$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1 + z + z^2 + z^3 + \dots)$$
provided $|z| < 1$ (this is just a geometric series). Therefore, by the uniqueness of power series, we get
$$f(z) = -\sum_{k=0}^{\infty} z^k.$$

(ii) Determine the power series representation for
$$f(z) = \frac{1}{(z-1)}$$
within the largest circle centered at $z = i$.

First note that a convergent power series for $f(z)$ centered at $z = i$ will exist provided $|z - i| < |1 - i| = \sqrt{2}$. Next we observe that

$$\frac{1}{z-1} = \frac{-1}{(1-i)-(z-i)} = -\frac{1}{(1-i)} \cdot \frac{1}{1-\frac{z-i}{1-i}}$$
$$= \frac{1}{i-1} \sum_{k=0}^{\infty} \left(\frac{z-i}{1-i}\right)^k$$

which converges as a geometric series provided

$$\left|\frac{z-i}{1-i}\right| < 1$$

or $|z - i| < \sqrt{2}$.

2. Uniqueness of Power Series

We now consider further generalizations of the results we proved for entire functions to functions which are analytic in a disc.

Proposition 2.1. *If f is analytic at a, then*

$$g(z) = \begin{cases} \frac{f(z)-f(\alpha)}{z-\alpha} & z \neq \alpha \\ f'(\alpha) & z = \alpha \end{cases}$$

is also analytic at α.

Proof. First observe that since f is analytic at $z = \alpha$, then it has a power series representation in some disc containing α i.e.

$$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f''(\alpha)}{2!}(z-\alpha)^2 + \dots$$

But then in that same neighbourhood, we have

$$g(z) = f'(\alpha) + \frac{f''(\alpha)}{2!}(z-\alpha) + \frac{f^{(3)}(\alpha)}{3!}(z-\alpha)^2 + \dots$$

Hence since $g(z)$ is equal to convergent power series in that same disc, it must be analytic at α.

□

Proposition 2.2. *If $f(z)$ is analytic at α then it is infinitely differentiable at α.*

Proof. By definition, a function is analytic at α if it is analytic in a region containing α. But then it follows that f can be represented by a power series in a disc containing α. However, this completes the proof since power series are infinitely differentiable.

□

Next we generalize the uniqueness theorem for power series.

Theorem 2.3. *(The Uniqueness Theorem for Power Series) Suppose that f is analytic in a region D and that $f(z_n) = 0$ where $\{z_n\}$ is a sequence of distinct points and $z_n \to z_0 \in D$. Then $f(z) = 0$ for all $z \in D$.*

Proof. Let C be any disc contained in D and centered at $z = \alpha$. Then a power series for $f(z)$ exists at $z = \alpha$ and by the uniqueness of power series it follows that $f = 0$ throughout C. Therefore, we just need to consider the points of D which cannot be covered by a disc which is fully contained in C and is centered at α. The proof we use is similar to an earlier idea.
Define

$$A = \{z \in D | z \text{ is a limit of zeros of of } f\}$$

and

$$B = \{z \in D | z \notin A\}$$

Clearly $A \cap B = \emptyset$. We shall show that both A and B are open and it will follow that $A = D$ since D is a region and we know $A \neq \emptyset$.
The set A is open by the uniqueness theorem for power series. Specifically, if $a \in A$, then there is an open disc containing a in which $f(z)$ has a power series representation which will be 0 (since a is a limit of zeros), so every point in that disc will also be a limit of zeros.
Now suppose that $b \in B$. Since b is not a limit point of zeros, there must exist some δ such that $|z - b| < \delta$ implies z is not a zero of $f(z)$. Clearly every point in this disc is also an element of B and hence B is open. The result follows.

□

The following result is immediate.

Corollary 2.4. *If two functions f and g are analytic in a region D and agree on a set with an accumulation point in D then $f = g$ throughout D.*

Proof. This is easily proved by considering the function $f - g$. □

The results we have proved can be used to determine conditions for when an entire function is a polynomial.

Theorem 2.5. *If f is entire and $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial.*

Proof. First we observe that $f(z)$ has a finite number of zeros. To see this, observe that since $f \to \infty$, there is some $M > 0$ such that $|f(z)| > 1$ for $|z| > M$. It follows that the only zeros of f occur in the disc $D(0, M)$. Since this set is bounded, if there are an infinite number of zeros, there will be an accumulation point of zeros and hence by the uniqueness $f(z) = 0$ everywhere (since it is entire).

Next let $\alpha_1, \dots, \alpha_N$ denote the zeros of $f(z)$ and define

$$g(z) = \frac{f(z)}{(z-\alpha_1)\dots(z-\alpha_N)}$$

Since $f(z)$ is entire, this function is also entire (as proved in the last chapter) and it is never 0 since we divided out by the zeros and hence

$$h(z) = \frac{1}{g(z)} = \frac{(z-\alpha_1)\dots(z-\alpha_N)}{f(z)}$$

is also entire. Since $f(z) \to \infty$ as $z \to \infty$ it follows that $|h(z)| \to 0$ as $z \to \infty$. In particular, $|h(z)| \leqslant A$ for some fixed constant A, so by Liouvilles theorem, $h(z) = k$ for k a constant. It follows that

$$f(z) = k(z-\alpha_1)\dots(z-\alpha_N),$$

so we are done.

□

3. The Mean Value Theorem and the Maximum Modulus Principle

We finish by considering more generalizations from results of real analysis. We start with an extension of the mean value theorem.

Theorem 3.1. *If f is analytic in D and $\alpha \in D$, then*

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\vartheta})d\vartheta$$

for any disc $D(\alpha, r)$ contained in D.

Proof. This is simply a reformulation of the Cauchy integral formula taking the parameterization around the disc $D(\alpha, r)$ as $z = \alpha + re^{i\vartheta}$.

□

Definition 3.2. We call α a relative maximum of $f(z)$ is $|f(\alpha)| \geqslant f(z)$ for all z in some disc centered at α. Likewise, we define a relative minumum of $f(z)$.

The following result shows that an analytic function cannot attain relative minimums and maximums.

Theorem 3.3. *(The Maximum Modulus Principle) A non-constant analytic function in a region D does not have any interior maximum points.*

Proof. We need to show that for a fixed $z \in D$, given any $\delta > 0$ we can find some $w \in D(z, \delta)$ such that $|f(w)| > |f(z)|$. Suppose that $C \subset D$ is a circle of radius r centered at z parameterized by $z(\vartheta) = z + re^{i\vartheta}$ and let M be the maximum value of $f(z)$ on C. The mean value theorem implies

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\vartheta})d\vartheta$$

so it follows that

$$|f(z)| \leqslant \left|\int_0^{2\pi} f(z + re^{i\vartheta})d\vartheta\right| \leqslant M$$

by the ML-formula. It follows that on any circle C centered at z, there exists $z_0 \in C$ such that $|f(z)| \leqslant M = f(z_0)$. To show that $f(z)$ cannot be a maximum value, we need to show that equality cannot hold - that is, for each circle C containing z, there cannot exists a point z_0 on C with $|f(z)| = M = f(z_0)$.
Suppose this is the case. Then on any circle C we have

$$M = |f(z)| = \left|\int_0^{2\pi} f(z + re^{i\vartheta})d\vartheta\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z + re^{i\vartheta})|d\vartheta = M$$

so it follows that $|f(z + re^{i\vartheta})| = M$ for all ϑ i.e. $|f|$ is constant on the circle C. Since this is true for all circles containing z within D, and so it follows that f is constant (since it is analytic) which is contrary to our assumptions. Thus there must exist $w \in D$ such that $|f(z)| < |f(w)|$ i.e. f attains no maximum value in D.

□

The maximum modulus principle can be restated as follows.

Corollary 3.4. *If $f(z)$ is analytic on a region D and defined on the boundary ∂D, then the maximum value of $f(z)$ on the closed region $\bar{D}$ is attained on ∂D.*

In an identical way, we can determine a minimum modulus principle.

Theorem 3.5. *If f is a non-constant analytic function in a region D then no point $z \in D$ can be a relative minimum of f unless $f(z) = 0$.*

Proof. Suppose that $f(z) \neq 0$. Then we can apply the maximum modulus principle to the function $g = 1/f$ to get the result. □

Homework:
Questions from pages 78-79; 1,3,5,6,8,9,11