Simply Connected Domains

"Generalizing the Closed Curve Theorem"

We have shown that if $f(z)$ is analytic inside and on a closed curve C, then
$$\int_C f(z)dz = 0.$$
We have also seen examples where $f(z)$ is analytic on the curve C, but not inside the curve C and
$$\int_C f(z)dz \neq 0$$
(for example $f(z) = 1/z$ over the unit circle centered at 1. In this section, we want to consider exactly which types of regions the closed curve theorem holds and over which types of regions it does not hold.

1. The General Cauchy Closed Curve Theorem

In order to describe the most general region in which the Cauchy integral formula holds, we need to introduce some new topological definitions. The first new idea we introduce is that of a simply connected region. To help understand this new concept, we shall introduce two equivalent definitions (one which considers the properties of a region and one which considers properties of the complement).

Warning. For a region to be simply connected, in the very least it must be a region i.e. an open, connected set.

Definition 1.1. A region D is said to be simply connected if any simple closed curve which lies entirely in D can be pulled to a single point in D (a curve is called simple if it has no self intersections).

Definition 1.2. A region D is simply connected if for any $z \in D^c$ (the complenent of D) and $\varepsilon > 0$, there is a continuous curve $\gamma(t)$ with $0 \leqslant t < \infty$ such that

(i) $d(\gamma(t), D^c) < \varepsilon$ for all $t \geqslant 0$

(ii) $\gamma(0) = z_0$

(iii) $\lim_{t\to\infty} \gamma(t) = \infty$

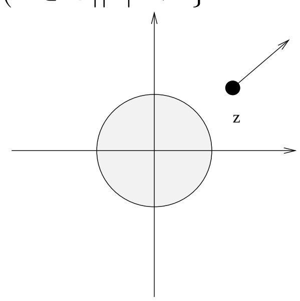
Note that by the second definition, a sufficient condition for a region D to be simply connected is that given any point z_0 in the complement, there is a smooth curve connecting z_0 to ∞ which lies entirely within D^c. It should be noted however that this is only a sufficient condition and not a necessary condition i.e. there exists simply connected regions whose complements do not satisfy this.

To illustrate this concept, we consider a number of different examples. We shall use both the naive definition and the formal definition to prove whether each given region is simply connected.

Example 1.3. Determine with reasons which of the following regions are simply connected.

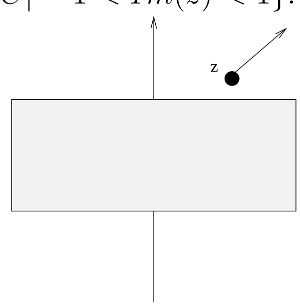
(i) The unit disc $\{(z \in \mathbb{C} || z| \leqslant 1\}$ including the boundary.
This is not an open set, so is not a region and hence cannot be a simply connected region.

(ii) The unit disc $\{(z \in \mathbb{C} || z| < 1\}$.



The set is a region. Using the first definition, clearly it is simply connected because if we place any loop in D, it can be pulled to a point. Using the second definition, we can connect any point z in D^c to ∞ by taking a radial line from z outward i.e. $\gamma(t) = z + zt$.

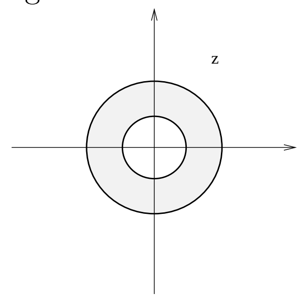
(iii) The strip $\{z \in C | -1 < Im(z) < 1\}$.



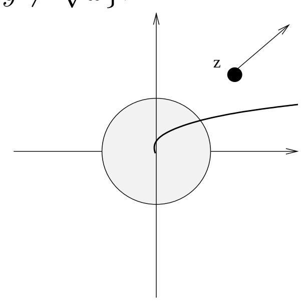
The set is a region. Using the first definition, clearly it is simply connected because if we place any loop in D, it can be pulled to a point. Using the second definition, we can connect any point z in D^c to ∞ by taking a radial line from z outward like last time i.e. $\gamma(t) = z + zt$.

(iv) The region D which consists of the disc of radius 2 minus the disc of radius 1 i.e. the set $\{z \in \mathbb{C} | 1 < |z| < 2\}$.
This set is clearly not simply connected. For the first definition, take any loop which loops around the unit disc. This loop cannot be pulled to a point and hence D is not simply connected. For the second definition, observe that any point in the interior of the disc is in D^c and clearly there is no path

from any point in the interior of the unit disc to ∞ which remains in the region D^c.



(v) The interior of the unit disc minus the curve $y = \sqrt{x}$ i.e. $\{z \in \mathbb{C} | |z| < 1, y \neq \sqrt{x}\}$.



The set is a region. Using the first definition, clearly it is simply connected because if we place any loop in D, it can be pulled to a point. Using the second definition, we can connect any point z in D^c to ∞ by taking a radial line from z outward like last time unless it lies on the curve $y = \sqrt{x}$. However, in this case we can define our curve along the line $y = \sqrt{x}$ and then radially out from the point of intersection between the disc and the line (which we call z_0) i.e.

$$\gamma(t) = \begin{cases} z + t + \sqrt{t}i & 0 \leqslant t \leqslant a \\ z_0 + z_0 t & t > a \end{cases}$$

where a is the value such that $z + a + \sqrt{a}i = z_0$.

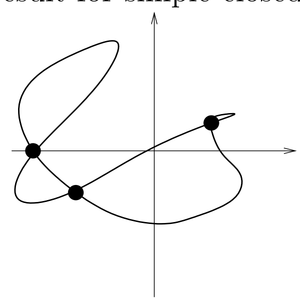
(vi) The complex plane minus the origin i.e. $\{z \in \mathbb{C} | |z| > 0\}$.

This is not simply connected. Any loop which contains $z = 0$ cannot be pulled to a point using the first definition. For the second definition, observe that D^c consists of just the point $z = 0$, so there is no way to determine a path from $z = 0$ to ∞ which stays in D^c.

Our next task is to show that the integral theorem and all related results hold for any function which is analytic in some simply connected region for any curve in that region.In order to prove this result, we some additional facts.

Lemma 1.4. *Suppose C is a closed curve contained in a simply connected region D. Then the interior of C is contained in D (by the interior, we mean the finite region with boundary C).*

Proof. Suppose that C is some closed curve. Then C may intersect itself a number of times. However, if this happens, then C may be broken up into a number of simple closed curves (see illustration). Thus it suffices to prove the result for simple closed curves.



Assume C is a simple closed curve and that there are points in the interior of C which are not included in the region D. Then it is impossible to contract C to a point within D, so D cannot be simply connected which is contrary to our assumption. This it follows that the interior of C is included in D.
Alternatively, if there exists a point z_0 in the interior of C which is not in D, then since any path from z_0 to ∞ will have to pass through C, the second definition of simply connected would imply D is not simply connected, so the result follows.

□

Lemma 1.5. *Suppose that A is a compact and $\{U_i\}$ is a set of open sets with*

$$A \subset \cup U_i.$$

Then there exists a finite subset $\{U_j\}$ such that

$$A \subset \cup U_i.$$

Proof. Homework.

□

We are now ready to prove the main result. The actual proof is much more technical than we shall consider,so we shall instead consider a sketch proof of the result. The main idea stems back to the results we have already developed.

Theorem 1.6. *Suppose $f(z)$ is analytic in a simply connected region D and that C is a smooth closed curve contained in D. Then*

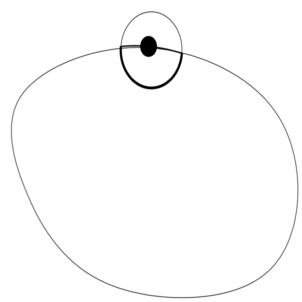
$$\int_C f(z)dz = 0.$$

Proof. First note that by the remarks made in the last lemma, it suffices to prove this result for simple closed curves, so assume that C is simple. Let R denote the interior region of C. Suppose that z_1 is a point on C. Since D is a region, we can find an open ball B centered at z_1 so that $B \subset D$. It follows that $f(z)$ is analytic in B and by the closed curve theorem,

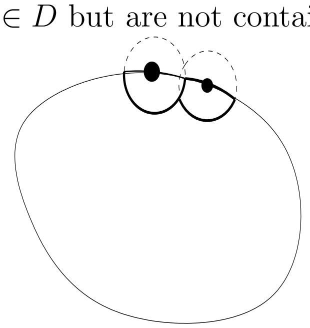
$$\int_\Gamma f(z)dz = 0$$

for any closed curve contained in B. In particular, if we take Γ to be the curve which which consists of the boundary of C contained in B and the points on the boundary of B with $z \in D$ (see illustration), we get

$$\int_\Gamma f(z)dz = 0.$$



Let $\{B_i\}$ denote the set of all balls around each point on C which are contained in D where B_i is centered around $z_i \in C$. Since C is a compact subset of $\mathbb{C}$, there is a finite subset of open balls $\{B_j\}_{j=1}^n$ which covers C centered at the points $z_j \in C$. Without loss of generality, suppose that the points z_j are ordered on the curve numerically. Starting at z_1, we construct a closed curve Γ_1 as described above. Next, we move to the next ball B_2 and construct a closed curve Γ_2 which consists of the boundary of C contained in B_2 which is not in B_1 and the points on the boundary of B_2 with $z \in D$ but are not contained in B_1 (see illustration). We continue this process until we reach the final ball B_n where we construct a closed curve Γ_n which consists of the boundary of C contained in B_n which is not in B_1 or B_{n-1} and the points on the boundary of B_n with $z \in D$ but are not contained in B_1 or B_{n-1}.



Now observe that the boundaries of the balls B_j in the interior of the region R form a finite number of closed curves within the region R (since there are only finitely many Γ_i). Since D is simply connected we have $R \subset D$ and so

$$\int_{\Pi} f(z)dz = 0$$

for any closed curve Π in R. In particular, if $\{\Pi_i\}_{i=1}^{m}$ denotes the set of closed curves which consist of the boundaries of the B_j in D, then

$$\int_{\Pi_i} f(z)dz = 0$$

for each i. Finally, by the way we have constructed the regions, after cancellation along all the curves contained inside D, we get

$$\int_C f(z)dz = \sum_{i=1}^{m} \int_{\Pi_i} f(z)dz + \sum_{j=1}^{n} \int_{\Pi_j} f(z)dz$$

and the result follows.

$\square$

The following is am immediate consequence of the closed curve theorem.

Theorem 1.7. *If f is analytic in a simply connected region D, there exists $F(z)$ such that $F'(z) = f(z)$ for all $z \in D$.*

Proof. Fix some $z_0 \in D$ and for any $z \in D$ define

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta.$$

Using the closed curve theorem, this is well defined. To show that $F' = f$, we use the fact that

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(\zeta)d\zeta$$

where $\int_z^{z+h}$ denotes the integral along the simplest polygonal path from z to $z+h$ and imitate the proof of the result for entire functions.

$\square$

We illustrate with an example.

Example 1.8. Suppose that C is a circle of radius r centered at α and $a \in \mathbb{C}$ but $a \notin C$ i.e. a does not lie in the boundary of C or in the interior of C. Then

$$\int_C \frac{1}{z-a}dz = 0$$

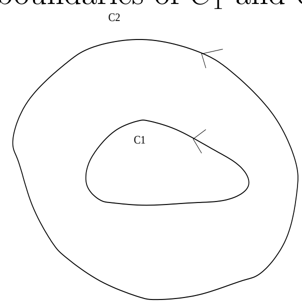
since the circle C is contained in a simply connected subset of the domain of analyticity of $1/(z-a)$.

One very useful consequence of the general closed curve theorem is the following.

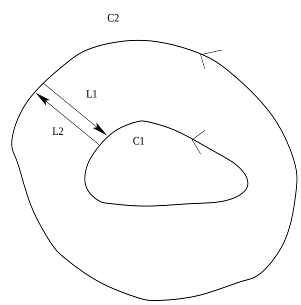
Proposition 1.9. *Suppose C_1 and C_2 are simple closed curves oriented in the same direction and that C_1 is contained in the interior of C_2. Then if $f(z)$ is analytic on the region bounded between C_1 and C_2 and on C_1 and C_2. Then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Proof. Let C_1 and C_2 be simple closed curves with C_1 contained in the interior of C_2 and suppose that $f(z)$ is analytic on C_1 and C_2 and in the region bounded by the boundaries of C_1 and C_2 as illustrated below.



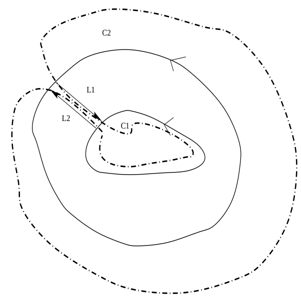
Next draw a line L (or curve) from C_2 to C_1 which does not intersect itself. We consider it as two different curves oriented in opposite directions - L_1 from C_2 to C_1 and L_2 from C_1 to C_2 (see illustration below).



Let $C = C_2 \cup L_1 \cup -C_1 \cup L_2$ and observe that since L_1 and L_2 are oriented in the opposite directions, we have

$$\int_C f(z)dz = \int_{C_2 \cup -C_1} f(z)dz.$$

Now observe that the curve C is contained in a simply connected region D - specifically the region bounded between C_1 and C_2 and the line L. To see this observe that if z is a point not in C_2, then clearly it can be connected to ∞ by a path fully contained in D^c. If z soes lie in D^c but inside C_2, then it either lies on the line L or in the interior of D, so we can connect z to infinity via a line which passes over L. Thus the region D (as illustrated below) is a simply connected region containing C.



By the general integral formula, since $f(z)$ is analytic inside and on C, it follows that

$$\int_C f(z)dz = \int_{C_2 \cup -C_1} f(z)dz = 0$$

or

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

as postulated.

□

2. The Analytic Function $\log(z)$

We finish by trying to define a logarithm. In single variable calculus, a logarithm function is easy to define since any exponential function is one-to-one. In complex analysis, the exponential function is not one-to-one, so we need to impose domain restrictions in order to define an inverse function. We define a logarithm as follows.

Definition 2.1. We say that $f(z)$ is an analytic branch of $\log(z)$ in a domain D if

(i) f is analytic in D

(ii) f is am inverse of $g(z) = e^z$ i.e. $e^f(z) = z$.

We first observe that since e^z is periodic with period $2\pi i$, if f is any any analytic branch of $\log(z)$, then so is the function $f(z) + 2\pi k i$ for any integer k. Next we observe the conditions our definition for a logarithm impose on the function $f(z)$.

Now suppose that $f(z) = u(z) + iv(z)$. If $z = Re^{i\vartheta}$, then since $e^{f(z)} = z$, using the exponential properties, we have

$$e^{u(z)+iv(z)} = e^{u(z)}e^{iv(z)} = Re^{i\vartheta}$$

so $e^{u(z)} = R$ and $v(z) = Arg(z) = \vartheta + 2k\pi$. Thus we can always find a branch of $\log(z)$ be setting $f(z) = \log(|z| + iArg(z)$ where $\log(|z|)$ is the real natural logarithim and $Arg(z)$ is the argument taken with $0 \leqslant Arg(z) < 2\pi$.

The major problem with the way we have defined this branch is that it is unclear whether it is analytic (or even how to go about showing it is analytic). Therefore, as can be done with the real logarithm, we can use integration to define a logarithm in complex variables.

Theorem 2.2. *Suppose that D is a simply connected domain and that $0 \notin D$. Choose $z_0 \in D$, fix a value of $\log(z_0)$ so that $e^{log(z_0)} = z_0$ and set*
$$f(z) = \int_{z_0}^{z} \frac{d\zeta}{\zeta}.$$
Then f is an analytic branch of $\log(z)$ in D.

Proof. First note that f is well defined since $1/\zeta$ is analytic in D and so the integral along any two paths are equal. Next, we observe that $f'(z) = 1/z$, so f is analytic in D (since it is differentiable). To show it is an analytic branch of $\log(z)$, we need to show that $e^{f(z)} = z$.
To see this, we observe that if $g(z) = ze^{-f(z)}$, then
$$g'(z) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0$$
and so g is constant. Next observe that
$$g(z) = g(z_0) = z_0 e^{-f(z_0)} = z_0 e^{-\log(z_0)} = z_0 \frac{1}{z_0} = 1.$$
Thus it follows that $ze^{-f(z)} = 1$ or
$$e^{f(z)} = z.$$
□

In an identical way, for any analytic function $f(z)$, we can define a branch of $\log(f(z))$ in an identical way. In particular, this provides us with a way to define analytic branches of different functions which in real analysis may have inverses, but in complex analyis do not. We illustrate with an example.

Example 2.3. For any analytic branch of $\log(z)$, we can define a branch of $z^{1/2}$ as
$$z^{1/2} = e^{\frac{1}{2}\log(z)}.$$
This really is a branch since
$$(e^{\frac{1}{2}\log(z)})^2 = e^{\log(z)} = z$$
so it provides solutions to the equation $w^2 = z$.
More specifically, if we chose $z_0 = 1$ and $\log(0) = 1$, an analytic branch of $\log(z)$ is
$$f(z) = \int_{0}^{z} \frac{d\zeta}{\zeta}.$$
It follows that a corresponding analytic branch of $z^{1/2}$ will be
$$g(z) = e^{\frac{1}{2}f(z)}.$$
Note that unlike the logarithm, there are only two branches of $z^{1/2}$. Specifically, given any branch $f(z)$ of $\log(z)$, any other branch will be of the form $f(z) + 2n\pi i$. However, since
$$e^{\frac{1}{2}f(z)} = e^{\frac{1}{2}(f(z)+2\pi ki]}$$

for any even integer k, there will only be two branches for $z^{1/2}$ depending upon whether k is even or odd.

Homework:
1) Suppose that A is a compact and $\{U_i\}$ is a set of open sets with

$$A \subset \cup U_i.$$

Show that there exists a finite subset $\{U_j\}$ such that

$$A \subset \cup U_i.$$

Questions from pages 101-102; 1,3,7,8,9