# Isolated Singularities and Laurent Series

"Functions which are not analytic at a point".

In this chapter we consider function which are analytic in the entire complex plane except at a finite number of points. We shall see that many of the results we have already developed will allow us to draw important conclusions about such functions. We shall also see how these results can be applied to problems such as real integrals of real functions and summation of real valued series.

## 1. Classification of Isolated Singularities

We start with some definitions and facts.

**Definition 1.1.** For a given $z_0 \in \mathbb{C}$, by a deleted neighbourhood of $z_0$, we mean an open set

$$\{z : 0 < |z - z_0| < d\}$$

for some fixed $d$.

**Definition 1.2.** A function $f(z)$ is said to have an isolated singularity at $z_0$ if $f$ is analytic in a deleted neighbourhood of $z_0$ but not at $z_0$.

Though it is beyond the scope of this course, in order to completely analyze isolated singularities, we shall need the following result.

**Theorem 1.3.** *Suppose $f$ is continuous in a open set $D$ and analytic in a deleted neighbourhood $N$ of $z_0 \in D$ with $N \subset D$. Then $f$ is analytic at $z_0$.*

The following result is immediate.

**Corollary 1.4.** *If $f$ has an isolated singularity at $z_0$, then it must be discontinuous at $z_0$.*

Our first task will be to describe all the different types of isolated singularity.First we note that singularities can be catagorized into the following three catagories.

**Definition 1.5.** Suppose $f$ has an isolated singularity at $z_0$.

- $(i)$ If there exists a function $g(z)$ which is analytic at $z_0$ and such that $g(z) = f(z)$ in a deleted neighbourhood of $z_0$, we say $f(z)$ has a removable singularity at $z_0$ (it can be made into an analytic function by simply changing the value at $z_0$).
- $(ii)$ If for $z \neq z_0$, $f$ can be written in the form $f(z) = A(z)/B(z)$ where $A$ and $B$ are analytic at $z_0$, $A(z_0) \neq 0$ and $B(z_0) = 0$, we say $f$ has a pole at $z_0$. If $B$ has a zero of multiplicity $k$ at $z_0$ we say $f$ has a pole of order $k$ at $z_0$.

($iii$) If $f$ has neither a removable singularity or a pole at $z_0$, we say it has an essential singularity.

**Example 1.6.** ($i$) The function $f(z) = \sin(z)/z$ has a removable singularity at $z = 0$. It is removable since $f(z)$ agrees with

$$g(z) = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!}$$

which is an everywhere convergent power series.

($ii$) The function $f(z) = 1/z^2$ has a pole of order 2 at $z = 0$.

($iii$) The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

We now analyze these three different possibilities. We start by giving criteria for determining what type a given singulaity is.

**Theorem 1.7.** *(Riemann's Principle) If $f$ has an isolated singularity at $z_0$ and if $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then the singularity is removable.*

*Proof.* Consider

$$h(z) = \begin{cases} (z - z_0) f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

By construction, $h(z)$ is continuous, and since $f(z)$ is analytic in a deleted neighbourhood of $z_0$, so is $h(z)$. It follows that $h(z)$ is also analytic at $z_0$. Let $g(z) = h(z)/(z - z_0)$. Clearly $g(z)$ is analytic in a deleted neighbourhood of $z_0$. However, since $h(z_0) = 0$, it follows that

$$g(z) = \frac{h(z)}{z - z_0}$$

is also analytic in the deleted neighbourhood and at $z_0$ (as proved in previous sections). But then $f(z) = g(z)$ for all $z \neq z_0$ in a deleted neighbourhood of $z_0$, so it follows that $f(z)$ has a removable singularity at $z_0$).

□

**Corollary 1.8.** *If $z_0$ is an isolated singularity of $f$ and $f$ is bounded in some neighbourhood of $z_0$, then $z_0$ is a removable singularity.*

**Theorem 1.9.** *If $z_0$ is an isolated singularity of $f(z)$ and there exists an integer $k$ such that*

$$\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0$$

*but*

$$\lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0,$$

*then $f$ has a pole of order $k$.*

*Proof.* We imitate the proof of the last result. Specifically, we set

$$g(z) = \begin{cases} (z - z_0)^{k+1} f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

Since $g$ is continuous in the whole neoghbourhood, it will be analytic in the whole neighbourhood. If we define $A(z) = g(z)/(z - z_0) = (z - z_0)^k f(z)$, then since $g(z_0) = 0$ it follows that $A(z)$ will also be analytic in the whole neighbourhood of $z_0$ and at the point $z_0$. Observe also that $A(z_0) = (z - z_0)^k \neq 0$ by assumption. Thus

$$f(z) = \frac{A(z)}{(z - z_0)^k}$$

for $z \neq z_0$ and $A(z_0) \neq 0$, so $f(z)$ has a poloe of order $k$ at $z_0$ by definition.

$\square$

Observe that in both of these cases, we have some form of control over the singularity - specifically, $f(z)$ differs from an analytic function by a multiple of $(z - z_0)$. The last case we examine are the essential singularities - points where such control is not possible.

**Theorem 1.10.** *(Casorati-Weierstraß Theorem) If $f(z)$ has an essential singularity at $z = z_0$ and if $N$ is a deleted neighbourhood of $z_0$, then the range $R\{f(z)| \in N\}$ is dense in the complex plane.*

*Proof.* Assume that $R$ is not dense in $\mathbb{C}$. Then there exists some disc $D = D(w, \delta)$ such that $f(z)$ takes no value in $D$ i.e. $|f(z) - w| > \delta$ for all $z \in D$. Now consider the function $h(z) = 1/(f(z) - w)$. Since $f(z)$ is analyirc inside $N$ (except at $z_0$) and since $f(z) \neq w$, it follows that $h(z)$ will be analytic inside $N$ except at $z_0$ (which will be a singularity). Also note that

$$|h(z)| = \frac{1}{|f(z) - w|} < \frac{1}{\delta}$$

for all $z \in N$, and in particular, it will be bounded. Hence $z_0$ will be a removable singularity of $h(z)$, so there exists an analytic function $g(z)$ which is analytic at $z_0$ and inside $N$ with $g(z) = h(z)$ for $z \neq z_0$. But then

$$f(z) = \frac{wg(z) + 1}{g(z)},$$

so either has a removable singularity at $z_0$ (if $g(z_0) \neq 0$) or a pole (if $g(z_0) = 0$ by definition of a pole). Hence the result.

$\square$

## 2. Laurent Expansions

In the last few chapters, we developed the theory of Taylor serier for analytic functions - infinite series which equal a given complex analytic function in a given disc. In this section we generalize this idea to isolated singularities - specifically, we shall determine a power series representation for a function $f(z)$ which has isolated singularities. We start with a definition and basic facts about so called "two sided series" or Laurent Series.

**Definition 2.1.** We say

$$\sum_{k=-\infty}^{\infty} \mu_k = L$$

if both

$$\sum_{k=0}^{\infty} \mu_k$$

and

$$\sum_{k=1}^{\infty} \mu_{-k}$$

both converge and sum to $L$.

**Theorem 2.2.**

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

*is convergent in the domain* $D = \{z : R_1 < |z| and |z| > R_2\}$ *where*

$$R_1 = \overline{\lim_{k\to\infty}} |a_{-k}|^{1/k}$$

*and*

$$R_2 = \frac{1}{\overline{\lim}_{k\to\infty} |a_k|^{1/k}}$$

*If* $R_1 < R_2$ *then* $D$ *is an annulus and* $f$ *is analytic in* $D$.

*Proof.* By the root test for regular power series,

$$f_1(z) = \sum_{k=0}^{\infty} a_k z^k$$

converges for $|z| < R_2$ and

$$g(w) = \sum_{k=1}^{\infty} a_{-k} \frac{1}{z^k} = \sum_{k=1}^{\infty} a_{-k} w^k$$

where $w = 1/z$ converges for $|w| < 1/R_1$ or $|z| > R_1$. Thus $f(z)$ converges on the intersection of these two regions. Also, since $g(w)$ and $f(z)$ are both power series, they are both analytic on their domains of convergence, and hence $f(z)$ will also be analytic on the intersection of these domains.

$\square$

We these preliminary results, we are now ready to prove that any function which is analytic in an annulus has a Laurent expansion in that annulus (thus generalizing Taylors Theorem for functions analytic in a disc).

**Theorem 2.3.** *If $f$ is analytic in the annulus $A : R_1 < |z| < R_2$, then $f$ has a unique Laurent expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

*in $A$ where*

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} dz$$

*where $C$ is any circle in $A$ centered at $z = 0$.*

*Proof.* The proof is similar to the case for Taylor series. Suppose $z \in A$ and choose $r_1$ and $r_2$ with $R_1 < r_1 < |z| < r_2 < R_2$ and let $C_1$ and $C_2$ denote the circles of radius $r_1$ and $r_2$ centered at 0 respectively. Now by our previous results, since $f$ is analytic in $A$, so is

$$g(w) = \frac{f(w) - f(z)}{w - z}.$$

It follows by the closed curve theorem (and our previous observations) that

$$\int_{C_2 - C_1} g(w) dw = 0$$

or

$$\int_{C_2 - C_1} \frac{f(w)}{w - z} dw = \int_{C_2 - C_1} \frac{f(z)}{w - z} dw.$$

Applying Cauchys Theorem, it follows that $\int_{C_1} f(z)/(w - z) dw = 2\pi i$ and $\int_{C_2} f(z)/(w - z) dw = 0$ (since $z$ does not lie in $C_2$). Thus we have

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z} dw - \int_{C_1} \frac{f(w)}{w - z} dw.$$

We analyze these two integrals individually.
On $C_2$, since $|w| > |z|$ (so $|z/w| < 1$), we have

$$\frac{1}{w - z} = \frac{1}{w(1 - z/w)} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots$$

and on $C_1$, since $|w| < |z|$ we have

$$\frac{1}{w - z} = -\frac{1}{z - w} = -\frac{1}{z} - \frac{w}{z^2} - \frac{w^2}{z^3} - \ldots$$

It follows that

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\left(\sum_{k=0}^{\infty}\frac{f(w)z^k}{w^{k+1}}\right)dw + \frac{1}{2\pi i}\int_{C_1}\left(\sum_{k=-1}^{-\infty}\frac{f(w)z^k}{w^{k+1}}\right)dw.$$

Next observe that if $C$ is any circle in $A$ centered at the origin then

$$\int_C \frac{f(w)z^k}{w^{k+1}}dw = \int_{C_1}\frac{f(w)z^k}{w^{k+1}}dw = \int_{C_2}\frac{f(w)z^k}{w^{k+1}}dw$$

(using a similar argument as the beginning of the proof applying the closed curve theorem since for every $k$ the function $f(w)/w^{k+1}$ is analytic), so we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\left(\sum_{k=0}^{\infty}\frac{f(w)z^k}{w^{k+1}}\right)dw + \frac{1}{2\pi i}\int_{C_1}\left(\sum_{k=-1}^{-\infty}\frac{f(w)z^k}{w^{k+1}}\right)dw = \sum_{k=-\infty}^{\infty} a_k z^k$$

where

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}dw$$

i.e. $f(z)$ has a power series representation at every point in $A$.
To see uniqueness, observe that if

$$\sum_{k=-\infty}^{\infty} a_k z^k$$

is any other power series representation for $f(z)$, then since $f(z)/z^{k+1}$ is analytic in $A$, for each $k$ and any circle $C$ in $A$ centered at 0 we have

$$\int_C \frac{f(z)}{z^{k+1}}dz = \int_C \frac{\sum_{n=-\infty}^{\infty} a_n z^n}{z^{k+1}}dz = \sum_{n=-\infty}^{\infty}\int_C a_n z^{n-k-1}dz = \int_C \frac{a_k}{z}dz$$

(since

$$\int_C a_n z^{n-k-1}dz = 0$$

for $n \neq k$ and $2\pi i$ for $n = k$). Thus we have

$$\int_C \frac{f(z)}{z^{k+1}}dz == \int_C \frac{a_k}{z}dz = 2\pi i a_k$$

or

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{k+1}}dz.$$

□

The following results are immediate.

**Corollary 2.4.** *If $f(z)$ is analytic in $R_1 < |z - z_0| < R_2$, then $f$ has a unique Laurent series representation*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

*where*

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}dz$$

*for $C = C(z_0, R)$ with $R_1 < R < R_2$.*

**Corollary 2.5.** *If $f$ has an isolated singularity at $z_0$, then for some $\delta > 0$ and $0 < |z - z_0| < \delta$,*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

*with the $a_k$ defined as above.*

Of course, in general we do not want to calculate the coefficients of a Laurent expansion explicitly using the results we have developed - instead we usually want to manipulate geometric series as we often did with Taylor series. We illstrate with some examples.

**Example 2.6.** (*i*) The function $f(z) = 1/z^2$ has an isolated singularity $z = 0$. Notice that around $z = 0$ the Laurent expansion is

$$\frac{1}{z^2}$$

i.e. $1/z^2$ is its own Laurent expansion.

(*ii*) The function $f(z) = 1/(z^2(z-1))$ has an isolated singularity at $z = 0$ and $z = 1$. We can consider the different power series representations around different annuluses.

Around $z = 0$, provided $|z| < 1$, we have

$$\frac{1}{z^2(z-1)} = \frac{1}{z^2}\frac{-1}{(1-z)} = -\frac{1}{z^2}(1 + z + z^2 + z^3 + \dots) = -\sum_{k=-2}^{\infty} z^k$$

Around $z = 0$, for $|z| > 1$, we have

$$\frac{1}{z^2(z-1)} = \frac{1}{z^2}\frac{\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{z^3}(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots) = \sum_{k=-\infty}^{-3} z^k.$$

Around $z = 1$, provided $|z - 1| < 1$, we have

$$\frac{1}{z^2(z-1)} = \frac{1}{(1+(z-1))^2}\frac{1}{(z-1)} = \frac{1}{(z-1)}\left(\frac{1}{1+(z-1)}\right)^2)$$

$$= \frac{1}{(z-1)}\left(1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots\right)^2$$

$$= \frac{1}{z-1} - 2 + 3(z-1) - 4(z-1)^2 + \dots = \sum_{k=-1}^{\infty}(-1)^{k+1}(k+2)(z-1)^k$$

Around $z = 1$, for $|z-1| > 1$, we have

$$\frac{1}{z^2(z-1)} = \frac{1}{(z-1)^2}\left(\frac{1}{1+\frac{1}{(z-1)^2}}\right)^2 \frac{1}{z-1}$$

$$= \frac{1}{(z-1)^3}\left(1 - \frac{1}{(z-1)} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \dots\right)^2 = \sum_{k=-\infty}^{-3} (-1)^k \frac{2+k}{(z-1)^k}.$$

(*iii*) The function $\sin(1/z)$ has an essential singularity at $z = 0$. To see this, we observe that around $z = 0$, $\sin(z)$ has the power series representation

$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!},$$

so $\sin(1/z)$ has the power series representation

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{z^{2k+1}(2k+1)!}.$$

**Definition 2.7.** If

$$f(z) = \sum a_k (z - z_0)^k$$

is the Laurent expansion around an isolated singulairty $z_0$,

$$\sum_{-\infty}^{-1} a_k (z - z_0)^k$$

is called the principle part of $f$ at $z_0$ and

$$\sum_{0}^{\infty} a_k (z - z_0)^k$$

the analytic part.

The principle part of the Laurent expansion around an islolated singularity describes completely the type of singularity it is. Specifically, we have the following:

**Proposition 2.8.** *Suppose $z_0$ is an isolated singularity of $f(z)$.*

(*i*) *If $z_0$ is removable then all the coefficients of the principle part are* 0.

(*ii*) *If $z_0$ is a pole of order $k$ then $C_{-k} \neq 0$ but $C_{-N} = 0$ for all $N > k$.*

(*iii*) *If $z_0$ is an essential singularity then there are infinitely many nonzero coefficients in its principle part.*

*Proof.* (*i*) If $f(z)$ has a removable singularity at $z_0$, then since $f(z) = g(z)$ $(z \neq z_0)$ for some function analytic at $z_0$, the Laurent expansion of $f(z)$ must agree with the Taylor series of $g(z)$.

(*ii*) If $f(z)$ has a pole of order $k$, then $f(z) = A(z)/B(z)$ where $B(z)$ has a zero of order $k$. In particular, $f(z) = Q(z)/(z-z_0)^k$ where $Q(z)$ is some analytic function. Since $Q(z)$ is analytic at $z_0$, it has a Taylor expansion

$$Q(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $Q(z_0) \neq 0$. Thus

$$f(z) = \frac{Q(z)}{(z - z_0)^k} = \sum_{l=-k}^{\infty} C_l (z - z_0)^l$$

where $C_l = a_{l+k}$ and $C_{-k} \neq 0$.

(*iii*) If $f(z)$ has an essential singularity at $z_0$, then there is no $N$ such that $(z-z_0)^N f(z)$ is analytic (else it would be a pole), so there must be infinitely many nonzero terms in the principle part of $f(z)$.

□

As a corollary to our results, we can justify the fact that any rational function can be written as a sum of partial fractions.

**Theorem 2.9.** *(Partial Fraction Decomposition) Any proper rational function*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z - z_1)^k_1 (z - z_2)^{k_2} \ldots (z - z_n)^{k_n}}$$

*where $P$ and $Q$ are polynomials with deg$(P) <$ deg$(Q)$ can be expanded as a sum of polynomials in $1/(z - z_i)$ for $i = 1, 2, \ldots, n$.*

*Proof.* Since $R$ has a pole of order at most $k$ at $z = z_1$,

$$R(z) = P_1(\frac{1}{z - z_1}) + A_1(z)$$

where $P_1(1/(z - z_0))$ is the principle part of the Laurent expansion around $z_1$ and $A_1$ is the analytic part (so $P_1(1/(z - z_1))$ is a sum of (strictly positive) powers of $1/(z-z_1)$). It follows that $R(z)-P_1(1/(z-z_1))$ has a removable singularity at $z_1$ (since we have subtracted the principle part) and the same principle part as $R(z)$ at each of the singularities $z_2, \ldots, z_n$ (since $P_1(z/(z - z_1))$ will be analytic at each of the singularities $z_2, \ldots, z_n$).
We can repeat this process taking

$$A_i(z) = R(z) - P_1\left(\frac{1}{(z - z_1)}\right) - P_2\left(\frac{1}{(z - z_2)}\right) - \cdots - P_i\left(\frac{1}{(z - z_i)}\right)$$

by subtracting the principle part of the Laurent expansion of $R(z)$ from $A_{i-1}(z)$. This gives

$$A_n(z) = R(z) - P_1\left(\frac{1}{(z-z_1)}\right) - P_2\left(\frac{1}{(z-z_2)}\right) - \cdots - P_n\left(\frac{1}{(z-z_n)}\right)$$

Since all the singularities are removable, by defining the values of $A_n(z)$ at each of the points $z_1, \ldots, z_n$, it follows that $A_n(z)$ is an entire function. Notice also that since $R$ and its principle parts go to 0 as $z \to \infty$, it follows that $A_n(z)$ is bounded, and thus by Louvilles Theorem, it is constant, and in fact 0 (since it approaches 0 and $z \to \infty$. It follows that

$$R(z) = P_1\left(\frac{1}{(z-z_1)}\right) + P_2\left(\frac{1}{(z-z_2)}\right) + \cdots + P_n\left(\frac{1}{(z-z_n)}\right)$$

$\square$

**Homework**

Text Pages 113-114; Questions: 1, 3, 4, 5, 6, 7, 8, 9,