Homework

It is time to brush the dust off of your old calculus books and remember some techniques you will need in the coming weeks. The following problems cover the prerequisites and also some of the preliminaries we have just discussed.

(*i*) Does the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$$

exist? If so, find the limit and if not, explain why not.

(*ii*) Use the difference quotient definition of the derivative to find the derivative function of

$$f(x) = \frac{1}{x}.$$

(*iii*) Evaluate the following limits (with justification):

(a)

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

(b)

$$\lim_{n\to\infty} n^{\frac{1}{n}}$$

(c)

$$\lim_{n\to\infty} \frac{9^{n+1}}{10^n}$$

(*iv*) Write down parametric equations for the unit circle centered at the origin in the xy-plane oriented counter-clockwise.

(*v*) Convert the Euclidean coordinates $(1, 2)$ to polar coordinates.

(*vi*) Convert the Polar coordinates $(3, 2\pi/3)$ into Euclidean coordinates.

(*vii*) Determine whether the following series are (absolutely) convergent, conditionally convergent or divergent (with justification):

(a)

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

(b)

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$$

(c)

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$

(*viii*) Find the radius of convergence and interval of convergence of the following power series (with justification):

(a)
$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^2 5^n}$$

(b)
$$\sum_{n=1}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$$

(c)
$$\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$$

(*ix*) Determine which of the following curves are smooth:

(a) The space curve with parametric equations $x(t) = t^3$, $y(t) = t^4$, $z(t) = t^5$, $-1 \leqslant t \leqslant 1$.

(b) The 2-d curve with parametric equations $x(t) = 2\cos(t)$, $y(t) = 3\sin(t)$ with $0 \leqslant t \leqslant 2\pi$.

(*x*) Suppose C is the unit circle centered at the origin oriented counterclockwise. Evaluate the following line integrals (with justification):

(a)
$$\int_C xdx + ydy$$

(b)
$$\int_C -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

(*xi*) Derive the quadratic formula for the zeros of a quadratic $ax^2 + bx + c$ where $a \neq 0$.

(*xii*) Prove that $(1+x)^n \geqslant (1+nx)$, where n is a non-negative integer and x is any real number.

(*xiii*) Let $F = \{0, 1, 2, 3, 4\}$ and define operations of multiplication and addition on F by taking addition and multiplication on the integers modulo 5. Show that F is a field.

(*xiv*) Describe the following subset of $\mathbb{R}^3$:
$$\{x \in \mathbb{R}^3 || x| = 1\}$$

(*xv*) Suppose $x \in \mathbb{R}^2$ and $x \neq (0,0)$.

(a) What would have to be true of $y \in \mathbb{R}^2$ to guarantee that $x \cdot y = 0$.

(b) What would have to be true of $y \in \mathbb{R}^2$ to guarantee that $x \cdot y < 0$.