Introduction and Preliminaries

"Algebra and Calculus".

1. Introduction and Motivation

Loosely speaking, complex analysis is the subject which attempts to take all the ideas developed for real numbers and functions of real variables (the ideas from basic algebra, Calculus 1, Calculus 2, Vector Calculus) and generalize them to complex numbers and functions of complex variables. Some of these ideas will generalize directly (ideas of addition, multiplication etc.) and some will have very different generalizations (exponential functions, integrals, logarithms). Throughout the course, there will only be a small amount of completely new material developed (material which is unique to complex variables and has no analogue in real variables), so most of the ideas we develop should be familiar. Therefore, for most sections, our approach will be to first recall the real variable problem, and then discuss the complex variable analogue. Of course, it goes without saying that though many of the ideas we develop will not be new, most of the results we develop will be new because things quickly become very different when considering complex numbers.

The first obvious question to ask ourselves in complex analysis is why would we want to generalize the ideas of real numbers and functions of real variables to complex numbers and functions of complex variables? The answer to this is easy - complex analysis is one of the most useful branches of mathematics in the applied sciences. Though we shall only see a small number of applications, complex analysis reaches across disciplines such as fluid dynamics, electromagnetism, string theory, and quantum field theory. Complex analysis also has applications in branches of mathematics such as number theory, geometry and (real variable) calculus. For example, in Calculus 2, you probably once showed that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (HOW?). However, except using approximations, you were probably not able to determine exactly what it converged to. We shall use complex analysis to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. Prerequisites and Preliminaries

The prerequisites for this course include Calculus 1, Calculus 2, Vector Calculus, and discrete math. Though not necessary, it may be helpful to have seen some abstract algebra, real analysis and topology. Of course, not all of the material from these courses will be required, so below is a (non-exhaustive) list of some of the major ideas we shall need from the prerequisite courses.

(*i*) **Calculus 1** - continuity of a function (ε-δ definition), the definition of the derivative (difference quotient), differentiability of a function, limits (techniques and ε-δ).
(*ii*) **Calculus 2** - Parameterizing curves in 2-space, polar coordinates, sequences and series (definitions and techniques for showing convergence/divergence).
(*iii*) **Vector Calculus** - functions of two variables, space curves and parametrization of curves, definition of a line integral, basic line integral calculations (direct method, Green's Theorem and Fundamental Theorem of Calculus).
(*iv*) **Discrete Math** - Proof techniques (this course will be partially proof based - both in lectures and assignments, so proof techniques will be very important).

Though not from prerequisite courses, the following definitions are taken from courses some of you may have seen and will be required for the course.

Definition 2.1. Vector Calculus and Real Analysis A curve C given by parametric equations $(x(t), y(t))$ is called smooth if both derivatives $x'(t)$ and $y'(t)$ exist and $x'(t)$ and $y'(t)$ are not both equal to zero at the same point (except possibly at the endpoints of the curve).

Definition 2.2. Modern Algebra - A field is a triple $(F, +, *)$, where F is a nonempty set, and $+$ and $*$ are binary operations from $F \times F$ to F, such that:

(*i*) Both $+$ and $*$ are associative, i.e for all a, b, c in F, $a+(b+c) = (a+b)+c$ and $a*(b*c) = (a*b)*c$,
(*ii*) Both $+$ and * are commutative, i.e for all a, b belonging to F, $a+b = b+a$ and $a*b = b*a$,
(*iii*) The operation $*$ is distributive over the operation $+$ i.e for all a, b, c, belonging to F, $a*(b+c) = (a*b)+(a*c)$,
(*iv*) There exists an additive identity, i.e there exists an element 0 in F, such that for all a belonging to F, $a+0 = a$,
(*v*) There exists a multiplicative identity, i.e. there exists an element 1 in F different from 0, such that for all a belonging to F, $a*1 = a$,

(vi) For each element in F, there exists an additive inverse, i.e for every a belonging to F, there exists an element $-a$ in F, such that $a + (-a) = 0$,

(vii) For each nonzero element of F, there exists a multiplicative inverse, i.e for every $a \neq 0$ belonging to F, there exists an element a^{-1} in F, such that $a * a^{-1} = 1$.

All rules should be familiar from elementary arithmetic.

Example 2.3. (i) The set of rational numbers $\mathbb{Q}$ with the usual operations of addition and multiplication form a field.

(ii) The integers $\mathbb{Z}$ with the usual operations of addition and multiplication **do not** form a field (WHY?).

Definition 2.4. Linear Algebra and Real Analysis Let $\mathbb{R}$ denote the field of real numbers. For any non-negative integer n, let $\mathbb{R}^n$ denote the set of all n-tuples of real numbers. An element of $\mathbb{R}^n$ is written $x = (x_1, x_2, \ldots, x_n)$ where each x_i is a real number. We can define the following operations on $\mathbb{R}^n$:

(i) (Addition/Subtraction)

$$(x_1, x_2, \ldots, x_n) \pm (y_1, y_2, \ldots, y_n) = (x_1 \pm y_1, x_2 \pm y_2, \ldots, x_n \pm y_n)$$

(ii) (Scalar Multiplication)

$$c * (x_1, \ldots, x_n) = (c * x_1, \ldots, c * x_n)$$

(iii) (Inner Product)

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$$

(iv) (Norm)

$$|(x_1, x_2, \ldots, x_n)| = \Big((x_1, x_2, \ldots, x_n) \cdot (x_1, x_2, \ldots, x_n)\Big)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

The set $\mathbb{R}^n$ together with these operations is called Euclidean n-space.