

Complex Numbers and Complex Functions

1. THE FIELD OF COMPLEX NUMBERS

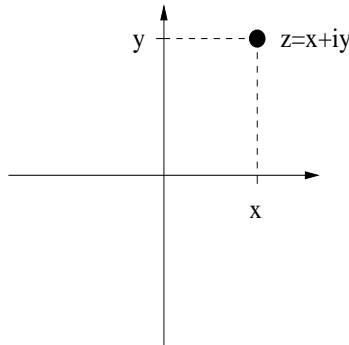
1.1. **Basic Definitions and Representations.** We start with some definitions.

Definition 1.1. We define the number i , called the **imaginary unit number** to be a number such that $i^2 = -1$.

Definition 1.2. A complex number is an expression of the form $z = x + iy$ where x and y are real numbers. We call x the **real part** of z , sometimes denoting it by $Re(z)$ and we call y the **imaginary part** of z sometimes denoting it by $Im(z)$. We call the set of all such numbers the set of **complex numbers** and denote it by \mathbb{C} .

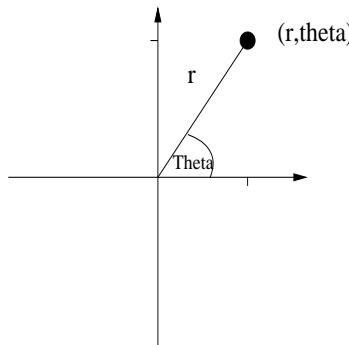
We can also represent complex numbers geometrically.

Definition 1.3. For a complex number $z = x + iy$, we represent it on a two dimensional coordinate system as the point in the plane with coordinates (x, y) . We call this geometric representation of \mathbb{C} the **complex plane**.



Since the complex numbers are represented on a plane, we can use polar coordinates as another representation for complex numbers. Such a representation will be useful for certain calculations.

Definition 1.4. Geometrically, we can represent the complex number $z = x + iy$ by the polar coordinates (r, ϑ) where $x = r \cos(\vartheta)$ and $y = r \sin(\vartheta)$. Algebraically, we have $z = x + iy = r \cos(\vartheta) + ir \sin(\vartheta)$.



Next we introduce a function called the polar exponential function which will give us a very useful tool for certain calculations with complex numbers.

Definition 1.5. We define the **polar exponential function** to be $e^{i\vartheta} = \cos(\vartheta) + i \sin(\vartheta)$. Note that $e^{i\vartheta}$ is a function of a real variable ϑ and the values of the function are evaluated by simply putting in the values of ϑ .

The polar exponential function satisfies a number of properties shared with the regular real exponential function. Many of these properties will be developed in detail when we consider the complex exponential function. However, for convenience in some of our initial work, the following properties will be useful.

Proposition 1.6. For any ϑ , ϑ_1 and ϑ_2 and positive integers n and m , the polar exponential function $e^{i\vartheta}$ satisfies the following properties:

- (i) $e^{i\vartheta_1} \cdot e^{i\vartheta_2} = e^{i(\vartheta_1 + \vartheta_2)}$
- (ii) $(e^{i\vartheta})^m = e^{im\vartheta}$
- (iii) $(e^{i\vartheta})^{(1/n)} = e^{i\vartheta/n}$

Proof. All the properties are simple consequences of trigonometric identities. For (i), we have

$$\begin{aligned} e^{i\vartheta_1} \cdot e^{i\vartheta_2} &= (\cos(\vartheta_1) + i \sin(\vartheta_1)) \cdot (\cos(\vartheta_2) + i \sin(\vartheta_2)) \\ &= (\cos(\vartheta_1) \cdot \cos(\vartheta_2) - \sin(\vartheta_1) \cdot \sin(\vartheta_2)) + i(\cos(\vartheta_1) \cdot \sin(\vartheta_2) + \sin(\vartheta_1) \cdot \cos(\vartheta_2)) \\ &= \cos(\vartheta_1 + \vartheta_2) + i \sin(\vartheta_1 + \vartheta_2) = e^{i(\vartheta_1 + \vartheta_2)} \end{aligned}$$

using the trigonometric sum formulas. From (i), (ii) can be derived using induction. Finally, (iii) can be proved using (ii) i.e. we consider $(\cos(\vartheta/n) + i(\sin(\vartheta/n))^n$. and show it is equal to

□

Remark 1.7. Note that if $z = r \cos(\vartheta) + ir \sin(\vartheta)$ is a complex number given in polar form, then it can always be written in terms of the polar exponential function. i.e. $z = r \cos(\vartheta) + ir \sin(\vartheta) = re^{i\vartheta}$.

Converting a complex number given in polar coordinates to an expression of the form $z = x + iy$ is very easy. We can also convert (in a fairly straight forward way) from $z = x + iy$ to polar coordinates. To do this, we need a couple of definitions.

Definition 1.8. We define the **modulus** of z , denoted $|z|$, to be the value $|z| = \sqrt{x^2 + y^2}$ and we define the **argument** of z , denoted $\arg(z)$ to be the angle ϑ made by the line connected z to the origin in \mathbb{C} and the positive x -axis.

Warning. The function $\arg(z)$ is a **multivalued function** i.e. there is not a unique value of $\arg(z)$ for any given z - it is unique only up to addition and subtraction of multiples of 2π . Usually, to make $\arg(z)$ a single valued function, we specify a domain of definition (for example $0 \leq \arg(z) < 2\pi$).

Fact 1.9. The polar coordinates of $z = x + iy$ are (r, ϑ) where $r = |z| = \sqrt{x^2 + y^2}$ and ϑ is a solution of $\tan(\vartheta) = \frac{y}{x}$ in the appropriate quadrant. We set the argument of $z = 0$ to be 0.

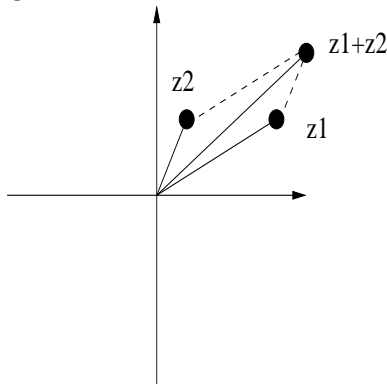
Example 1.10. Find all polar representations of the complex number $z = \sqrt{3} - i$.

We have $r = \sqrt{3 + 1} = 2$ and $\tan(\vartheta) = -1/\sqrt{3}$. Since z is in the fourth quadrant, it follows that $\vartheta = -\frac{\pi}{6}$. Thus a polar representation for z is $(2, -\frac{\pi}{6})$. Indeed, any polar representation for z will be $(2, -\frac{\pi}{6} + 2n\pi)$ for any $n \in \mathbb{Z}$.

1.2. The Field of Complex Numbers. Recall that a field is a set with two different operations which satisfy certain conditions. By extending the operations of addition and multiplication of real variables, we can turn the set of complex numbers into a field.

Definition 1.11. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we define the operations of addition and multiplication as follows:

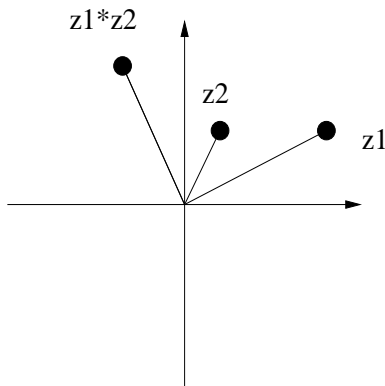
- (i) Algebraically, we define addition componentwise: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. Geometrically, this translated to the rule of adding vectors:



- (ii) Algebraically, we define multiplication treating i like a variable and using the fact that $i^2 = -1$:

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Geometrically, the multiple of z_1 and z_2 is the complex number with $|z| = |z_1||z_2|$ and $\arg z = \arg z_1 + \arg z_2$:



Proposition 1.12. *The set of complex numbers together with these operations makes the \mathbb{C} into a field.*

Proof. This will be completed in your homework. □

The geometric interpretation of addition is clear. The geometric interpretation of multiplication is easy to see using polar coordinates and the polar exponential function properties. Specifically, if $z_1 = r_1 e^{i\vartheta_1}$ and $z_2 = r_2 e^{i\vartheta_2}$, then

$$z = z_1 z_2 = r_1 r_2 e^{i(\vartheta_1 + \vartheta_2)}$$

which has modulus $|z_1||z_2| = r_1 r_2$ and argument $\arg(z) = \vartheta_1 + \vartheta_2 = \arg(z_1) + \arg(z_2)$.

One useful application of our observations above is in solving “pure” equations, or equations of the form $z^n = z_0$. We illustrate with an example.

Example 1.13. Find all solutions to the equation $z^4 = 1 + i$.

First, in polar form, we have $1 + i = \sqrt{2}e^{\pi i/4}$. Now suppose $z = re^{i\vartheta}$ is a solution to $z^4 = 1 + i = \sqrt{2}e^{\pi i/4}$. Then we have

$$z^4 = r^4 e^{4i\vartheta} = \sqrt{2}e^{\pi i/4}.$$

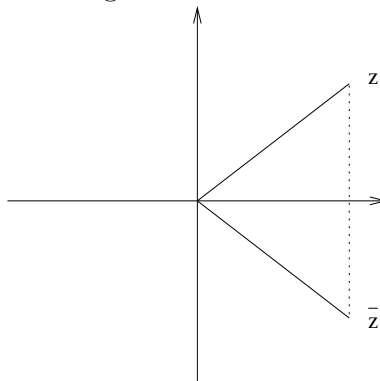
In particular, we must have $r^4 = \sqrt{2}$, so $r = \sqrt[8]{2}$ and $4\vartheta = \pi/4 + 2n\pi$ for every integer n . Thus we must have $\vartheta = \pi/16 + n\pi/2$. If we restrict to $0 \leq \vartheta \leq 2\pi$, we get the solutions $\vartheta = \pi/16, 9\pi/16, 17\pi/16, 25\pi/16$. Converting back to regular coordinates, we get the following four solutions:

$$1.069 + 2.13i, -2.13 + 1.069i, -1.069 - 2.13i, 2.13 - 1.069i.$$

In a similar fashion to our definitions for multiplication and addition, we can define subtraction and division of complex numbers. Subtraction is simple - however, given a quotient z_1/z_2 of complex numbers, it is not immediately clear how to write it in the form $x + iy$. To help with this problem, we introduce the following important idea.

Definition 1.14. For a given complex number $z = x + iy$, we define its complex conjugate, denoted \bar{z} to be the number $z = x - iy$.

Fact 1.15. Note that $|z| = |\bar{z}|$ and $\arg \bar{z} = -\arg z$ i.e. geometrically, \bar{z} is the reflection of z through the x -axis.



Proposition 1.16. For any real number z , $z\bar{z} = |z|^2$. In particular, $z\bar{z}$ is real.

Proof. The proof is easy. Suppose that $z = x + iy$. Then $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$. □

We can use the conjugate to rewrite any quotient, z_1/z_2 in the form $x + iy$. Specifically, we simply multiply by \bar{z}_2/\bar{z}_2 (which is the same as multiplying by 1) and algebraically simplify. We illustrate.

Example 1.17. Write $(2 + i)/(3 + i)$ in the form $a + bi$.

We have

$$\frac{2 + i}{3 + i} \cdot \frac{3 - i}{3 - i} = \frac{(6 + 1) + i(3 - 2)}{9 + 1} = \frac{7}{10} + \frac{i}{10}.$$

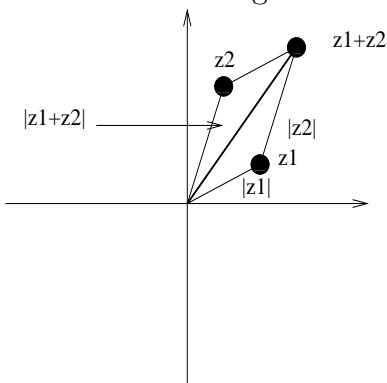
1.3. The Triangle Inequalities. We finish this section by considering one last important fact generalizes a result from vector calculus. It is a very important property of complex numbers and will be used extensively in many proofs throughout the course.

Proposition 1.18. (*The triangle inequalities*) For any complex numbers z_1 and z_2 , we have

$$||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$$

Proof. We can use either algebra or geometry to prove these statements. To illustrate the different techniques used, we shall prove one each way. We shall start by proving the second inequality using geometry (we shall use the fact that the second inequality holds to prove the first algebraically). Recall that the sum of two complex vectors is geometrically represented by the sum of the corresponding vectors. We

then construct a triangle and the result follows since it simply states that the lengths of the sums of the two sides of a triangle cannot exceed the length of the third side of the triangle.



Next we prove the first inequality algebraically. Fix z_1 and z_2 and without loss of generality, assume that $|z_1| \geq |z_2|$. Let $W_1 = z_1 + z_2$ and $W_2 = -z_2$. By the second inequality, we have

$$|W_1 + W_2| \leq |W_1| + |W_2|.$$

Substituting back in for z_1 and z_2 , we get

$$|z_1| \leq |z_1 + z_2| + |z_2| \text{ or } |z_1| - |z_2| \leq |z_1 + z_2|.$$

Since $|z_1| \geq |z_2|$, it follows that $||z_1| - |z_2|| \leq |z_1 + z_2|$, hence the result. \square

Homework

- (i) From the book, pages 7: Questions 1a, 1d, 2a, 2b, 2d, 3b, 3c, 3d, 4a, 4c, 4f
- (ii) Show that the set \mathbb{C} with the operations of addition and subtraction form a field.
- (iii) Prove the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ algebraically.