Relations and Equivalence Relations

In this section, we shall introduce a formal definition for the notion of a relation on a set. This is something we often take for granted in elementary algebra courses, but is a fundamental concept in mathematics i.e. the very notion of a function relies upon the definition of a relation. Following this, we shall discuss special types of relations on sets.

1. Binary Relations and Basic Definitions

We start with a formal definition of a relation on a set S.

Definition 1.1. A (binary) relation on a set S is a subset R of the Cartesian product $S \times S$. If R is a relation and $(x, y) \in R$, then we say "x is related to y by R" or simply xRy.

Example 1.2. The most familiar of all relations is the relation "$=$" (equals) which consists of all the elements $(x, x) \in S \times S$.

Example 1.3. Suppose that $S = \{0, 1, 2, 3\}$ and $R =<$ (less than with standard definition from the integers). Write down all the elements of R.

We have

$$R = \left\{ \begin{array}{ccc} (0,1), & (0,2), & (0,3), \\ (1,2), & (1,3), & (2,3) \end{array} \right\}$$

(since it consists of all elements (x, y) with $x < y$).

There are certain special properties a relation can have such as the following:

Definition 1.4. Suppose R is a relation on a set S. Then we define the following:

- We say R is reflexive if xRx for all $x \in S$
- We say that R is symmetric if xRy implies yRx for all $x, y \in S$
- We say R is transitive if xRy and yRz implies xRz for all $x, y, z \in S$

We illustrate with some examples.

Example 1.5. Show that the relation $<$ (less than) on $\mathbb{R}$ is a transitive relation which is not symmetric or reflexive.

Suppose $x < y$ and $y < z$. Then clearly $x < z$ and hence $<$ is transitive. We do not have $x < x$ and if $x < y$, then it is not the case that $y < x$, so it follows that it is neither reflexive or symmetric.

Example 1.6. Let $X = \{a, b, c\}$ and let $S = \mathcal{P}(X)$. We define a relation R on $\mathcal{P}(X)$ as follows: for all $A, B \in S$,

$$ARB \iff N(A) \leqslant N(B)$$

Determine whether R is reflexive, symmetric, transitive.

We check each property:

(i) Reflexive: Clearly this operation is reflexive since $N(A) \leqslant N(A)$

(ii) Symmetric: This operation is not symmetric. Specifically, take $A = \emptyset$ and $B = \{a\}$. Then $N(A) = 0 \leqslant N(B) = 1$. However, $N(B)$ is not less than or equal to $N(A)$ i.e. to be symmetric, we must have $N(B) \leqslant N(A)$ whenever $N(A) \leqslant N(B)$.

(iii) Transitive: This operation is transitive. Specifically, if $N(A) \leqslant N(B)$ and $N(B) \leqslant N(C)$, then $N(A) \leqslant N(C)$.

2. Equivalence Relations and Examples

A very important type of relation are the so-called equivalence relations defined as follows.

Definition 2.1. Suppose that R is a relation on a set S. Then we call R an equivalence relation if R is reflexive, symmetric and transitive.

We illustrate how to show a relation is an equivalence relation or how to show it is not an equivalence relation.

Example 2.2. Show that the relation D defined on $\mathbb{Z}$ by

$$xDy \iff 3|(x^2 - y^2)$$

is an equivalence relation.

We show each of the properties individually.

(i) Reflexive: For any $x \in \mathbb{Z}$, we have $x^2 - x^2 = 0$, and since $3|0$, it follows that xDx for all $x \in \mathbb{Z}$

(ii) Symmetric: Suppose xDy. Then $3|(x^2 - y^2)$ so $x^2 - y^2 = 3n$ for some $n \in \mathbb{Z}$. It follows that $y^2 - x^2 = 3(-n)$, and hence $3|(y^2 - x^2)$. Consequently yDx, so D is symmetric.

(iii) Transitive: Suppose xDy and yDz. Then there exists $n, m \in \mathbb{Z}$ such that $x^2 - y^2 = 3n$ and $y^2 - z^2 = 3m$. It follows that

$$x^2 - (3m + z^2) = 3n \text{ or } x^2 - z^2 = 3m + 3n = 3(n + m)$$

and so xDz i.e. D is transitive.

Since D is reflexive, symmetric and transitive, it follows that D is an equivalence relation.

Example 2.3. Suppose that X is a non-empty set and let $S = \mathcal{P}(X)$. Show that the relation for all $A, B \in S$,

$$ARB \iff N(A) \leqslant N(B)$$

is not an equivalence relation.

To show a relation is not an equivalence relation, we simply need to show that it is either not reflexive, symmetric or transitive. However, we already considered this example and show that it was not a symmetric relation. Specifically, take $A = \emptyset$ and $B = \{a\}$ where $a \in S$ (we are assuming S is non-empty). Then $N(A) = 0 \leqslant N(B) = 1$. However, $N(B)$ is not less than or equal to $N(A)$ i.e. to be symmetric, we must have $N(B) \leqslant N(A)$ whenever $N(A) \leqslant N(B)$. Thus this relation is not an equivalence relation.

3. Equivalence Classes and Partitions of Sets

An important application of equivalence relations is that they can be used to construct partitions of sets. To do this, we need the following definition.

Definition 3.1. Suppose R is an equivalence relation on S. Then for any $a \in S$, we define the equivalence class of a to be the set of all elements of S which are related to a by R. Symbolically, we write

$$[a] = \{x \in S | aRx\}$$

We illustrate with an example.

Example 3.2. Let $S = \{1, 2, 3, \ldots, 19, 20\}$ and define an equivalence relation R on S by

$$xRy \iff 4|(x - y)$$

Determine the equivalence classes of R.

In order to determine equivalence classes, we simply need to group the elements together according to which ones are equivalent to each other. We start with 1 and move onward:

$$[1] = \{1, 5, 9, 13, 17\}$$
$$[2] = \{2, 6, 10, 14, 18\}$$
$$[3] = \{3, 7, 11, 15, 19\}$$
$$[4] = \{4, 8, 12, 16, 20\}$$

Notice that any other equivalence class we construct will be the same as one of these e.g $[1] = [5]$. Thus we have found all the different equivalence classes of R.

Notice that the equivalence classes in the last example split up the set S into 4 mutually disjoint sets whose union was S. In particular, the equivalence classes formed a partition of S. This is in fact always true, and is a consequence of the following more general theorem.

Theorem 3.3. *Suppose that S is a set. Then:*

(i) *Suppose R is an equivalence relation of S. Then the classes of R form a partition of S.*

(ii) Suppose that $P = \{S_1, \ldots, S_k \ldots\}$ is a partition of the set S. Then we can define an equivalence relation on S by

$$xRy \iff x, y \in S_i \text{ for some } i$$

i.e. two elements are equivalent if they are in the same set.

Moreover, this correspondence between equivalence relations and partitions is a one-to-one correspondence i.e. the number of equivalence relations on a set is equal to the number of partitions of that set.

Proof. Suppose that R is an equivalence relation on S and let $S_1, S_2, \ldots$ denote the equivalence classes of S. In order to show that they form a partition, we need to show that they are mutually disjoint and the union of all is S.
To show that

$$\cup S_i = S$$

it suffices to show that every $s \in S$ lies in at least one of these sets. However, since R is an equivalence relation, given any $s \in s$, sRS, so $s \in [s]$ i.e. s will always be a member of its own equivalence class. Thus every element of S appears in at least one equivalence class.
To show they are disjoint, we shall show that if $y \in [x]$, then $[y] = [x]$. Suppose that $y \in [x]$. Then we have xRy, and hence yRx since R is symmetric. Now if $z \in [x]$, then we have xRz, and it follows that yRz (by transitivity), and hence $z \in [y]$. In particular, $[x] \subseteq [y]$. Using an identical argument, we can show $[y] \subseteq [x]$ and thus $[x] = [y]$. Hence the set of equivalence classes of R form a partition of S.
Now suppose that $S_1, \ldots$ are a partition of S and let R be the relation defined by

$$xRy \iff x, y \in S_i$$

We shall show this is an equivalence relation.
Clearly it is reflexive since xRx i.e x lies in the same subset as itself. Next, if xRy, then $x, y \in S_i$ for some fixed i, so $y, x \in S_i$ for some fixed i, and hence yRx, so R is symmetric. FInally, if xRy and yRz, then $x, y, z \in S_i$ for some fixed i. In particular, $x, z \in S_i$ and hence xRz, so R is transitive. It follows that R is an equivalence relation.
Clearly this construction define a one-to-one correspondence. □

We finish with a couple of examples.

Example 3.4. Determine the total number of equivalence relations on a set with three elements.

Since there is a one-to-one correspondence between partitions of sets and equivalence relations, it suffices to count the number of partitions of a set with three elements. Let $S = \{a, b, c\}$. Then the possible partitions are:

$$\{S\},$$

$$\{\{a\},\{b,c\}\},$$
$$\{\{b\},\{a,c\}\},$$
$$\{\{c\},\{a,b\}\},$$
$$\{\{a\},\{b\},\{c\}\}.$$

In particular, there are five different partitions on S and hence five different equivalence relations which can be imposed on S.

Homework

(i) From the book, pages 592-594 (Section 10.2): Questions: 12, 13, 25, 26, 30, 37

(ii) From the book, pages 608-610 (Section 10.3): Questions: 3, 7, 8, 12, 17, 23,