

Section 1.1: Logical Form and Logical Equivalence

An argument is a sequence of statements aimed at demonstrating the truth of an assertion. The assertion at the end of an argument is called the **conclusion**, and the preceding statements are called **premises**. The **content** an argument are the things the argument is claiming to prove things about and the **logical form** of the argument is the structure of the argument independent of content. Logic is the study of the logical form of arguments and a study of the validity of arguments - it does not determine whether or not the content of the argument is right or wrong, but rather allows us to analyze whether the truth of the conclusion of the argument necessarily follows from the truth of the premises. That is, if we assume the premises are true, we can use logic to determine whether or not the conclusion necessarily follows. In this section we shall study how we can identify the type of logical form of a given statement and determine the truth values of different statements.

1. LOGICAL FORM

We shall first study the problem of identifying the logical form of a statement. For this, we need a few initial definitions.

Definition 1.1. A statement (or proposition) is a sentence that is true or false, but not both. The truth value of a statement is whether it is true or false.

Example 1.2. Which of the following are statements:

- (i) The moon is round (Statement - either true or false, can't be both)
- (ii) It is raining outside (Not statement - depends upon where you are)
- (iii) $2 + 2 = 5$ (Statement - always false)
- (iv) $x + y = 3$ (Not statement - depends upon values of x and y)

The content of a statement is usually found in the smaller statements which it can be broken down into. To make an statement independent of its content, and hence to allow us to study logical form, we usually denote a statement by a variable, say p or q . Statements in logic are often built up from other statements using logical connectives - such statements are usually called compound statements. Our next task is to consider consider the three elementary logical connectives and their truth values depending upon the truth value of the statements between connectives.

Definition 1.3. If p is a statement, the **negation** of p is “not p ” or “It is not the case that p ” and is denoted $\sim p$. It has opposite truth value from p : if p is true, $\sim p$ is false, and if p is false, $\sim p$ is true.

Definition 1.4. If p and q are statements, the **conjunction** of p and q is “ p and q ” denoted $p \wedge q$. If either p or q is false, or both are false, then $p \wedge q$ is false (so $p \wedge q$ is true only when p and q are both true).

Definition 1.5. If p and q are statements, the **disjunction** of p and q is “ p or q ” denoted $p \vee q$. If either p or q is true, or both are true, then $p \vee q$ is true (so $p \vee q$ is false only when p and q are both false).

Before we look at some examples, we need to make a couple of observations.

Remark 1.6. Note that “or” can be interpreted in two different ways: either as we have interpreted it (called the inclusive “or”), or it could be interpreted as true only when either p or q are true, but not both (the exclusive “or”). In logic, we only ever use the inclusive “or” as defined above.

Remark 1.7. The word “but” also often occurs in logical argumentation in the place of “and”. Therefore, in a logical statement where “but” occurs, we make the following translation:

$$\begin{array}{lcl} p \text{ but } q & \text{means} & p \text{ and } q \\ \text{neither } p \text{ nor } q & \text{means} & \sim p \text{ and } \sim q \end{array}$$

We can now formally define what we mean by the logical form of a statement.

Definition 1.8. A statement form (or propositional form, or logical form) is an expression made up of statement variables, called component statements, (such as p , q , and r), and logical connectives (such as \sim , \vee and \wedge) that becomes a statement when actual statements are substituted for the component statement variables.

To identify the logical form of a statement, we need to remove all the content in any component statements. To do this, we can do the following:

- (i) Identify all the component statements which are not built up using connectives and give them names as variables (this removes content).
- (ii) Identify all the logical connectives connecting the statements.
- (iii) Rewrite the statement in symbolic form replacing the statements with variables and connectives with the connective symbols.

By taking these steps, we can identify the logical form of our statement. Our next task is to discuss the truth value of a compound statement,

but before we do this, we illustrate with a couple of examples how to identify logical form.

Example 1.9. Write down the logical form of the following statements.

- (i) This table is square but it is not pink

We have to be a little careful with this statement since technically the second component statement is not really a statement (“it is not pink” is not a statement since it depends upon what it is talking about). In this instance however, it is clear that we are talking about the same table, so we can simplify the sentence and preserve its meaning as follows:

This table is square but this table is not pink

We can now find the logical form of the statement:

p := the table is square

q := the table is pink

The logical form of this statement is $\sim q \wedge p$.

- (ii) Either it is not Monday today or I am wearing Wellington boots

We can now find the logical form of the statement:

p := it is Monday

q := I am wearing Wellington boots

The logical form of this statement is $\sim p \vee q$.

Remark 1.10. With reference to the first example, as a general case, logic and the rules of logic only apply to complete statements and not fragments of statements. Thus in order to determine logical form, occasionally we must rewrite the statement so all statements within it are complete statements.

2. EVALUATING THE TRUTH OF A LOGICAL FORM

The truth value of a logical form will depend on the form itself and the truth values of the component statements. To determine the truth value of a logical form, we construct a **truth table**. A truth table is simply a table of “T”’s (True) and “F”’s (False) which will tell us the truth value of a logical form depending upon the truth values of its component statements. We illustrate by constructing the truth tables for the three elementary connectives:

Example 2.1. The following are the truth tables for the three elementary connectives:

(i)

p	$\sim p$
T	F
F	T

	p	q	$p \wedge q$
	T	T	T
(ii)	T	F	F
	F	T	F
	F	F	F
	p	q	$p \vee q$
	T	T	T
(iii)	T	F	T
	F	T	T
	F	F	F

To determine truth values of more complicated logical forms, we can also use truth tables. The laws of logic are very similar to the laws of addition - to evaluate a long complicated logical form, you start by evaluating forms in the inner-most parentheses and then work outward. To make our lives a little easier, we usually add columns to evaluate each connective. We illustrate through a couple of examples.

Example 2.2. Determine the truth tables for the following statements:

- (i) $(p \vee q) \wedge \sim (p \wedge q)$ (note that this is the exclusive “or”, sometimes denoted “XOR”).

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

- (ii) $(p \vee \sim q) \wedge r$ (p or not q , and r)

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	F

3. LOGICAL EQUIVALENCES

We have developed the basic tools to analyze the truth values of logical forms. We shall now consider the case when two statements are considered the same.

Definition 3.1. Two statement forms are said to be logically equivalent if and only if they have identical truth values for any possible substitution of statements into their component statements. If P and Q are two equivalent logical forms, then we write $P \equiv Q$.

We can now state what we mean by two statements having the same logical form.

Definition 3.2. Two statements are said to be logically equivalent if their statement forms are logically equivalent.

Showing logical equivalence or inequivalence is easy. Two forms are equivalent if and only if they have the same truth values, so we construct a table for each and compare the truth values (the last column). If the columns are identical, the columns will be the same. Else they will be different.

Warning. When testing for equivalence or non-equivalence, make sure in the rows of each table the truth values of the variables match up (else your answer may not be right!)

Example 3.3. Show the following two statements are logically equivalent: $p \wedge (q \vee \sim q)$ and $(p \wedge q) \vee (p \wedge \sim q)$.

p	q	$p \wedge q$	$p \wedge \sim q$	$(p \wedge q) \vee (p \wedge \sim q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

p	q	$q \vee \sim q$	$p \wedge (q \vee \sim q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	T	F

Since the last two columns are the same, these two statements are the same.

Example 3.4. Show the following two statements are logically inequivalent: $\sim p \vee q$ and $p \wedge \sim q$.

p	q	$\sim p \vee q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \wedge \sim q$
T	T	F
T	F	T
F	T	F
F	F	F

As we have seen, testing whether two logical forms are equivalent, though it may take a long time, is a fairly easy process. This process can be used to develop “distributive laws” for logical connectives. Specifically, we have the following:

Theorem 3.5. (*De Morgan’s Laws*)

- (i) *The negation of an “and” statement is equivalent to an “or” statement in which each component is negated. In symbols:*

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

- (ii) *The negation of an “or” statement is equivalent to an “and” statement in which each component is negated. In symbols:*

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

There are many other logical equivalences which can be developed which are in some respects generalizations of the laws of arithmetic. For a summary of these equivalences, see the table on Page 14. Note that these laws can often be used to dramatically simplify logical forms and can often be used to prove logical equivalences without the use of truth tables. We illustrate how to use De Morgan’s laws and the other laws with a couple of examples.

Example 3.6. Use De Morgan’s laws to negate the following statements.

- (i) “I like mathematics and I like computer science”

If

p := I like mathematics

q := I like computer science

then the logical form of this statement is $p \wedge q$. Using De Morgan’s law, the negation will be $\sim p \vee \sim q$ translating to “either I do not like mathematics or I do not like computer science”.

- (ii) “Either that clock is slow or this class is dragging on”

If

p := that clock is slow

q := this class is dragging on

then the logical form of this statement is $p \vee q$. Using De Morgan’s law, the negation will be $\sim p \wedge \sim q$ translating to “that clock is not slow and this class is not dragging on”.

4. TAUTOLOGIES AND CONTRADICTIONS

There are two very special statements in mathematics - statements which are always true and statements which are always false. Since they are so important, we give them their own names.

Definition 4.1. A **tautology** is a logical form that is always true regardless of the truth values of the component statements. A statement whose form is a tautology is called a tautological statement.

Definition 4.2. A **contradiction** is a logical form that is always false regardless of the truth values of the component statements. A statement whose form is a contradiction is called a contradictory statement.

We illustrate.

Example 4.3. (i) Show that the statement $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

(ii) Show that the statement $p \vee \sim p$ is a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

(iii) Show that if c is a contradiction, then $p \vee c \equiv p$.

p	c	$p \vee c$
T	F	T
F	F	F

Note that we did not have any rows where c had truth value T since it is a contradiction (so always false).

We finish with one last example where we can use logical equivalences to simplify statements.

Example 4.4. Show that

$$\sim (p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p.$$

We shall show every step and state the laws we use:

$$\sim (p \vee \sim q) \vee (\sim p \wedge \sim q)$$

(De Morgan's Law)

$$\equiv \sim (p \vee \sim q) \vee \sim (p \vee q)$$

(De Morgan's Law)

$$\equiv \sim [(p \vee \sim q) \wedge (p \vee q)]$$

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(Distribution)

$$\equiv \sim [p \vee (\sim q \wedge q)]$$

(Negation Laws)

$$\equiv \sim [p \vee c]$$

(Identity Laws)

$$\equiv \sim p$$

Homework

- (i) From the book, pages 15-17: Questions: 2, 5, 8, 11, 17, 20, 22, 24, 32, 38, 43, 45, 50