

Section 1.3: Valid and Invalid Arguments

Now we have developed the basic language of logic, we shall start to consider how logic can be used to determine whether or not a given argument is valid. In order to do this, we shall first formally define exactly what we mean by an argument and then discuss different valid and invalid types of argument and how to distinguish between them

1. THE DEFINITION OF A VALID AND INVALID ARGUMENT

We start with the definition of an argument.

Definition 1.1. An argument (form) is a sequence of statements (forms). All statements (forms) in an argument (form) except for the final one, are called premises (or assumptions, or hypothesis). The final statement (form) is called the conclusion. The symbol $\circ\circ$ which is read “therefore” is normally placed just before the conclusion.

Now we have a formal definition for an argument, we can state what we mean by a valid argument.

Definition 1.2. An argument form is valid if whenever true statements are substituted in for the statement variables the conclusions is always true. To say an argument is invalid means that it is not valid.

The main point regarding a valid argument is that it follows from the logical form itself and has nothing to do with the content. When a conclusion is reached using a valid argument, we say the conclusion is **inferred** or **deduced** from the premises. Before we consider examples, we shall briefly examine how one can tell if a given argument form is valid or invalid.

Result 1.3. To test whether or not an argument is valid, we do the following:

- (i) Identify the premises and the conclusion
- (ii) Construct a truth table showing the truth values of the premises and the conclusion
- (iii) Look for all the rows where the premises are all true - we call such rows **critical rows**. If the conclusion is false in a critical row, then the argument is invalid. Otherwise, the argument is valid (since the conclusion is always true when the premises are true).

We illustrate with a couple of examples.

Example 1.4. Determine whether the following arguments are valid.

(i)

$$\begin{array}{l}
 p \rightarrow q \\
 q \rightarrow r \\
 \circ\circ p \vee q \rightarrow r
 \end{array}$$

Constructing a truth table, we have:

	p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \vee q$	$p \vee q \rightarrow r$
\rightarrow	T	T	T	T	T	T	T
	T	T	F	T	F	T	F
	T	F	T	F	T	T	T
	T	F	F	F	T	T	F
\rightarrow	F	T	T	T	T	T	T
	F	T	F	T	F	T	F
\rightarrow	F	F	T	T	T	F	T
\rightarrow	F	F	F	T	T	F	T

To help, we mark the critical rows. Notice that all critical rows have a true conclusion and thus the argument is valid.

(ii)

$$\begin{array}{l}
 p \vee q \\
 p \rightarrow \sim q \\
 p \rightarrow r \\
 \circ\circ r
 \end{array}$$

Constructing a truth table, we have:

	p	q	r	$p \vee q$	$p \rightarrow \sim q$	$p \rightarrow r$	r
	T	T	T	T	F	T	T
	T	T	F	T	F	F	F
\rightarrow	T	F	T	T	T	T	T
	T	F	F	T	T	F	F
\rightarrow	F	T	T	T	T	T	T
\rightarrow	F	T	F	T	T	T	F
	F	F	T	F	T	T	T
	F	F	F	F	T	T	F

To help, we mark the critical rows. Notice that the 6th row is a critical row with a false conclusion, so it follows that the the argument is invalid.

Warning. In logic, the words “true” and “valid” have very different meanings - truth is talking about the statements making up an argument and validity is talking about whether the conclusion follows from the premises. Note that a perfectly valid argument may have a false conclusion depending upon the truth value of the premises. Likewise, an invalid argument may have a true conclusion depending upon the truth value of the premises.

Example 1.5. As we noted above, the argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \circ\circ p \vee q \rightarrow r \end{array}$$

is a perfectly valid argument. Let $p :=$ “I sleep a lot”, $q :=$ “I don’t do any homework” and $r :=$ “I will do well in this class”. Then this translates to:

“If I sleep a lot, then I don’t do any homework. If I don’t do any homework, then I will do well in the class. Therefore, if I sleep a lot or don’t do any homework, I will do well in the class”.

As noted above, this is a perfectly valid argument, but clearly not a true conclusion! This is because though the first hypothesis is true, the second hypothesis is false (and hence the conclusion is false - see the truth table).

We finish with one more example of translating an argument into logical form and then testing validity.

Example 1.6. Determine the validity of the following argument:

“Robbery was the motive for the crime only if the victim had money in his pockets. But robbery or vengeance was the motive for the crime. Therefore, vengeance must have been the motive for the crime.”

Let $p :=$ “robbery was the motive for the crime”, $q :=$ “the victim had money in his pockets”, and $r :=$ “vengeance was the motive for the crime”. Then the argument translates as follows:

$$\begin{array}{l} p \rightarrow q \\ p \vee r \\ \circ\circ r \end{array}$$

The truth table is:

	p	q	r	$p \rightarrow q$	$p \vee r$	r
\rightarrow	T	T	T	T	T	T
\rightarrow	T	T	F	T	T	F
	T	F	T	F	T	T
	T	F	F	F	T	F
\rightarrow	F	T	T	T	T	T
	F	T	F	T	F	F
\rightarrow	F	F	T	T	T	T
	F	F	F	T	F	F

This is clearly not a valid argument - as stated above, if the victim had money in their pockets, and the motivation of the crime was robbery but not vengeance, this satisfies all hypothesis, but not the conclusion as suggested by the truth table.

2. DIFFERENT TYPES OF VALID ARGUMENT

We shall now consider some standard valid argument forms. A valid argument is sometimes called a **rule of inference** since the conclusion can always be inferred from the hypothesis. We shall list the valid arguments and most of the time we will not prove validity since it is usually fairly obvious. Once they have all been stated, we shall consider some examples of how to use these arguments.

Result 2.1. (Modus Ponens and Modus Tollens) Suppose p and q are statement forms. Then the following are valid arguments:

(i) The argument called **modus ponens** defined as

$$\begin{array}{c} p \rightarrow q \\ p \\ \circ\circ q \end{array}$$

(ii) The argument called **modus tollens** defined as

$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \circ\circ \sim p \end{array}$$

Proof. We shall show that modus tollens is valid.

p	q	$p \rightarrow q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
\rightarrow F	F	T	T	T

In this case there is only one critical row to consider, and its truth value is true. Hence this is a valid argument. □

Result 2.2. (Generalization) Suppose p and q are statement forms. Then the following arguments (called generalization) are valid:

$$\begin{array}{cc} p & q \\ \circ\circ p \vee q & \circ\circ p \vee q \end{array}$$

Result 2.3. (Conjunction) Suppose p and q are statement forms. Then the following argument (called conjunction) is valid:

$$\begin{array}{c} p \\ q \\ \circ\circ p \wedge q \end{array}$$

Result 2.4. (Specialization) Suppose p and q are statement forms. Then the following arguments (called specialization) are valid:

$$\begin{array}{cc} p \wedge q & p \wedge q \\ \circ\circ p & \circ\circ q \end{array}$$

Result 2.5. (Elimination) Suppose p and q are statement forms. Then the following arguments (called elimination) are valid:

$$\begin{array}{l} p \vee q \quad p \vee q \\ \sim p \quad \sim q \\ \circ\circ q \quad \circ\circ p \end{array}$$

Result 2.6. (Transitivity) Suppose p , q and r are statement forms. Then the following argument (called transitivity) is valid:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \circ\circ p \rightarrow r \end{array}$$

Result 2.7. (Proof by Division into Cases) Suppose p , q and r are statement forms. Then the following argument (called proof by division into cases) is valid:

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \circ\circ r \end{array}$$

Result 2.8. (Contradiction) Suppose p is statement form and let c denote a contradiction. Then the following argument (called proof by contradiction) is valid:

$$\begin{array}{l} \sim p \rightarrow c \\ \circ\circ p \end{array}$$

That is, if you can show that the hypothesis that p is false leads to a contradiction, then p has to be true.

Proof. The truth table for this argument is as follows:

\rightarrow	p	c	$\sim p \rightarrow c$	p
	T	F	T	T
	F	F	F	F

This is a valid argument - there is only one critical row and this row has a positive truth value. □

Before we consider some examples, we make a few remarks about these rules of inference:

- The names of the rules of inference we have described above usually describe exactly what the rule of inference does. For example, the “Elimination” rule eliminates one of the possible variable statements given that one of them has to be true, and we know one of them is not true.

- Contradiction is a very important tool in mathematics and is used for many important proofs. However, the idea of contradiction is marred in controversy since it supposes an important problem in mathematics (or more generally, logic) called “the law of the excluded middle”. Specifically, the law of the excluded middle states that “either p or not p ”. The main problem with the law of the excluded middle is that it can be used to prove the existence of mathematical objects without ever actually constructing one. A movement of logicians and mathematicians called the intuitionists believe this is an invalid argument because, unless one can actually construct a mathematical object, then how can one prove that it exists.

We finish by considering a couple of examples.

Example 2.9. You are on an island with two tribes of people - one tribe, called the Truth Tribe, always tell the truth, and the other, called the Lie Tribe, always lie. Two natives, A and B approach you and they say the following:

A: Both of us are from the Truth Tribe

B: A is from the Lie Tribe

Who is from which tribe?

We shall prove this by contradiction. Assume A is in the Truth Tribe. Then it follows that both A and B are in the Truth Tribe (since A always tells the truth). However, this would mean B always tells the truth, and thus A must be in the Lie Tribe. This means A is in the Lie Tribe and not in the Truth Tribe which is a contradiction. Thus our initial assumption must be wrong, so by contradiction, A is in the Lie Tribe.

Now assume B is in the Lie Tribe. Then since B always lies, A must be in the truth tribe, but this contradicts our earlier analysis. Thus, by contradiction, B must be in the Truth Tribe.

Example 2.10. Deduce the conclusion of the argument from the hypothesis explaining each step.

- (i) $\sim p \vee q \rightarrow r$
- (ii) $s \vee \sim q$
- (iii) $\sim t$
- (iv) $p \rightarrow t$
- (v) $\sim p \wedge r \rightarrow \sim s$
- (vi) $\circ\circ \sim q$

We reason as following:

- (i) Use Modus Tollens on (iii) and (iv) to conclude $\sim p$
- (ii) Using Generalization on $\sim p$ from our last deduction, we have $\sim p \vee q$

- (iii) We can now use Modus Ponens on the last statement and (i) to conclude r
- (iv) Since we know $\sim p$ and r , we can use conjunction to get $\sim p \wedge r$
- (v) Now we can use Modus Ponens on the last observation and (v) concluding $\sim s$
- (vi) Finally, we can use elimination on (ii). i.e. we know $s \vee \sim q$ and we know $\sim s$, therefore we must have $\sim q$.

3. INVALID ARGUMENTS OR FALLACIES

To finish, we look at some typical invalid arguments. A fallacy is an error in reasoning which leads to an invalid argument. Though there are far too many fallacies for us to list them all, three of the more common ones are as follows (you will have probably heard many arguments such as these coming from politicians and lawyers!):

- Using ambiguous premises and treating them as if they are unambiguous e.g. Cutting people is a crime. Surgeons cut people. Therefore, surgeons are criminals.
- Begging the question, or assuming what is to be proved without deriving it from the premises e.g. All lawyers always tell the truth. Jack is a lawyer. Therefore, Jack always tells the truth - this assumes that all Lawyers always tell the truth, and so assumes the conclusion.
- Jumping to a conclusion without adequate grounds e.g. Smoking cigarettes does not cause brain damage. Therefore, smoking cigarettes is healthy.

Another two common errors arising from the conditional statement are the following:

Example 3.1. (Converse Error) One common fallacy is the following:

$$\begin{array}{c} p \rightarrow q \\ q \\ \circ \circ p \end{array}$$

On first glance, it looks like Modus Ponens, but this is not since the second hypothesis is q and not p . This argument is invalid since we know p implies q , but just because q has occurred does not imply that p has occurred. The truth table of this argument is as follows:

\rightarrow	p	q	$p \rightarrow q$	q	p
	T	T	T	T	T
	T	F	F	F	T
\rightarrow	F	T	T	T	F
	F	F	T	F	F

This argument is invalid since the third row is a critical row with a false conclusion. An example of this argument would be something like:

If I run I will get there quicker

I got there quicker

Therefore I must have ran

Of course, this argument is false since it doesn't take into consideration that instead of running, I could have driven to get there quicker!

Example 3.2. (Inverse Error) One common fallacy is the following:

$$\begin{array}{l} p \rightarrow q \\ \sim p \\ \circ\circ \sim q \end{array}$$

On first glance, it looks like Modus Tollens, but this is not since the second hypothesis is $\sim p$ and not $\sim q$. This argument is invalid since we know p implies q , but just because p has not occurred does not imply that q has not occurred. As with the last example, we could show this argument is invalid using a truth table. An example of this argument would be something like:

If I run I will get there quicker

I did not run

Therefore I did not get there quicker

Again, this argument is false since it doesn't take into consideration that instead of running, I could have driven to get there quicker!

Homework

- (i) From the book, pages 41-43: Questions: 1, 4, 7, 11, 13b, 18, 20, 22, 28, 29, 31, 32, 33, 34, 37, 38c, 42