Section 2.1: Introduction to the Logic of Quantified Statements

In the previous chapter, we studied a branch of logic called "propositional logic" or "propositional calculus". Loosely speaking, propositional calculus is a "formal system" in which formulae representing propositions can be formed by combining atomic propositions using logical connectives, and a system of formal proof rules allows certain formulae to be established as "theorems", or valid arguments. The problem with propositional calculus is that it does not allow us to determine the validity of many arguments encountered both in every day situations or, more importantly, in mathematics. Take for example the following argument:

All primes are odd
3 is a prime
$\therefore$ 3 is odd

Clearly this is an invalid argument since one of the hypothesis is not true. However, it is impossible to break this up into an argument in propositional calculus because it cannot be broken down into simple statements and logical connectives (indeed, there are no logical connectives). Since such arguments are commonplace in mathematics we need to build upon traditional propositional logic and introduce further operations. The extended system of logic we shall consider in the next two sections is usually called "predicate calculus" or "predicate logic" since it involves the symbolic manipulation of predicates (an expression that can be true of something such as "is yellow") and quantified statements (statements which claim certain qualities of all or some of a type of object. e.g. "all men are mortal").

1. Basic Definitions

We start by developing the basic definitions needed for predicate calculus. We start with a formal definition of a predicate.

Definition 1.1. A predicate is a sentence that contains a finite number of variables which becomes a statement as soon as specific values are substituted in for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variable.

As with propositional logic, we usually use symbols to denote predicates and the variables in predicates. We illustrate through an explicit example.

Example 1.2. Consider the sentence "that person has brown hair". Clearly this is not a statement since its truth cannot be determined without knowing who "that person" refers to. Let P denote the predicate "has brown hair" (we call P a predicate symbol since it denotes a predicate - these are nearly always upper case). Then we define $P(x)$ to be the sentence "x has yellow hair" where x is called a predicate variable and is assumed to take values from a specific set (we shall take the domain of x to be all people in this room). Though this is not a statement (since it may be true or false, depending upon the value of x), as soon as we choose as specific value of x, this does become a statement. For example $P(I)$:="I have brown hair" is false, however, for other choices, it may be a true statement.

Since predicate calculus relies upon a small amount of set theory, we shall briefly introduce some common definitions and terminology which will be used throughout the semester. There are some typical domains of definition of certain functions and some standard notation for set inclusion. Some of the more common domains are:

- $\mathbb{R}$ denotes all real numbers
- $\mathbb{Z}$ denotes all integers
- $\mathbb{Q}$ denotes all rational numbers
- $\mathbb{N}$ denotes the natural numbers (the integers greater or equal to zero
- We sometimes use $\mathbb{R}^+$ to denote all non-negative real numbers and $\mathbb{R}^-$ to denote all non-positive real numbers, and likewise with the other sets defined above. Note that zero is considered neither positive or negative, so lies in both $\mathbb{R}^+$and $\mathbb{R}^-$

Common notation encountered in mathematics is the following:

- If x is an element of the set A, we write $x \in A$
- If x is not an element of A, we write $x \notin A$
- One way to write down a set is to write down all its elements in a set of braces. e.g. the set containing 1, 2 and 3 is written $\{1, 2, 3\}$
- The disadvantage to the previous notation is that it does not always work for infinite sets. However, for certain infinite sets where, from just a few elements it is clear what the rest of the set is, we use the brace notation defined above listing a few elements followed by three dots $\ldots$. e.g $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

In logic, of course, we are interested in the truth values of a given statement. Therefore, we need the following definition.

Definition 1.3. If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is all the elements of D that make $P(x)$ true when they are substituted for X. The truth set is denoted by

$$\{x \in D | P(x)\}$$

which is read "the set of all x in D such that $P(x)$".

Example 1.4. Suppose that $Q(x)$ be the predicate "$x^3 < 28$.

(*i*) Write $Q(4)$, $Q(-2)$ and $Q(0)$ and indicate which are true and which are false.
We have:
$Q(4) =$ "$4^3 < 28$ which is clearly not true
$Q(-2) =$ "$(-2)^3 = -8 < 28$ which is true
$Q(0) =$ "$0^3 < 28$ which is true

(*ii*) Determine the truth set of $Q(x)$ for $x \in \mathbb{N}$.
We are looking for all the natural numbers with the property that its cube is less than 28. Thus the truth set of this predicate is $\{0, 1, 2, 3\}$.

(*iii*) Determine the truth set of $Q(x)$ for $x \in \mathbb{Z}$.
We are looking for all the integers with the property that its cube is less than 28. The cube of any negative integer is negative so will be less than 28. The remaining integers are the natural numbers, and we found all natural numbers for which this predicate is true above. Thus the truth set of this predicate is $\{\ldots, -3, -2, -1, 0, 1, 2, 3\}$.

2. The Universal and Existential Quantifiers

One way to turn a predicate into a statement is to insist it is true of all the values in the domain of the variable. Likewise, another way to turn a predicate into a statement is to insist it is true of at least one value in the domain of the variable. In both cases, note that there is still a variable in the predicate - only now it is a true or false statement since it is referring to either all the elements in the domain or at least one element in the domain.
The universal quantifier, denoted $\forall$ is used to denote the phrase "for all" in a sentence with a predicate and the existential quantifier, denoted $\exists$ is used to denote the phrase "there exists" in a sentence with a predicate. For example, "All men are mortal" would be written "$\forall x$ where $x \in M$, x is mortal" where M denotes the set of all men, and "There exists a man who is mortal" would be written "$\exists x$ where $x \in M$, x is mortal" where M denotes the set of all men. These are the symbols used to turn a predicate into a statement as described above. The truth values and formal definitions of such statements are as follows:

Definition 2.1. Let $Q(x)$ be a predicate and D the domain of x.

(*i*) A universal statement is a statement of the form "$\forall x \in D, Q(x)$" and is read, "for all x in the domain D, $Q(x)$ is true". It is defined to be true if and only if for every value of x in D substituted into $Q(x)$, $Q(x)$ is true. It is false if there exists at least one x in D for which $Q(x)$ is false, and such an x is called a counterexample.

(*ii*) An existential statement is a statement of the form "$\exists x \in D, Q(x)$" and is read, "there exists x in the domain D, such that $Q(x)$ is true". It is defined to be true if and only if there is some value of x in D which when substituted into $Q(x)$, $Q(x)$ is true. It is false if there for every value of x in D, when substituted into $Q(x)$, $Q(x)$ is false.

We illustrate with a couple of examples.

Example 2.2. Let $D = \{0, 1, 2, 4\}$. Determine whether the following statements are true explaining your answer:

(i) $\forall x \in \mathbb{R}, x > 1/x$

Observe that this is not true for $x = 1/2$, and hence we have exhibited a value of x for which this statement is not true. It follows that this statement must be false.

(ii) $\forall x \in D, x^2 + 1 \geqslant 1$

We can check that each element of D against this property: $0^2 + 1 = 1 \geqslant 1$, $1^2 + 1 = 2 \geqslant 1$, $2^2 + 1 = 5 \geqslant 1$, and $4^2 + 1 = 17 \geqslant 1$. Since all elements of D satisfy this property, it follows that this statement is true.

(iii) $\exists x \in \mathbb{R}, x^2 = x$

Observe that $0^2 = 0$, so there does exists x in $\mathbb{R}$ with $x^2 = x$, and hence this statement is true.

(iv) $\exists x \in D, x^2 \geqslant 27$

We can check that each element of D against this property: $0^2 = 0 < 27$, $1^2 = 1 < 271$, $2^2 = 4 < 27$, and $4^2 = 16 < 27$. It follows that no elements of D satisfy this property and hence the statement is false.

3. Formal Versus Informal Lanuage

One of the very important techniques for succeeding in both logic and mathematics is being able to translate between the formal language of logic and the informal language we use in everyday conversation. The reason we need to be able to do this is twofold. First, most problems we encounter will be usually explain in regular informal language, and in order to determine truth values or validity, it is necessary to convert it to formal language. Alternatively, given a statement in formal language, sometimes it is much easier to interpret its real meaning when it is expressed in informal language (since this is how we communicate). We illustrate with a couple of examples.

Example 3.1. Translate the following statements in mathematics to equivalent informal statements.

(i) $\forall x \in \mathbb{Z}, 2x$ is even

Some of the things this translates to are:

- Any integer multiplied by 2 is even
- Every integer doubled is even

- For any integer x, if x is doubled, then the result is an even number

(*ii*) $\exists x \in \mathbb{R}, x^3 = x^2$

Some of the things this translates to are:

- There is some real number whose cube is itself
- Some real numbers cubed equal themselves
- There is at least one real number whose cube is itself

Example 3.2. Translate the following informal statements into equivalent formal statements.

(*i*) "Someone from class told me the homework was hard"

Let $P :=$"told me that the homework was hard" and let D be the set of all people from class. Then the above sentence translates to $\exists x \in D, P(x)$.

(*ii*) "Everyone who has taken this class says it is a difficult class"

Let $P :=$"says it is a difficult class" and let D be the set of all people who have taken this class. Then the above sentence translates to $\forall x \in D, P(x)$.

4. Universal Conditional Statements

One of the most important statements in mathematics is the universal conditional statement which combines a universal statement and a conditional statement. Specifically, a universal conditional statement is a statement of the form

$$\forall x, \text{ if } P(x) \text{ then } Q(x)$$

and reads "for all x, if $P(x)$ is true, then $Q(x)$ is true". The importance of this statement simply arises from the fact that if a given universal statement is true, then we are saying that any x with property P necessarily also has property Q. We illustrate with a couple of examples.

Example 4.1. Translate the following universal conditional statement $\forall x \in \mathbb{Z}$, if $|x| < 1$ then $-1 < x < 1$ to equivalent informal statements. Some of the things this translates to are:

- If a real number has absolute value less than 1, then it is between -1 and 1
- Any real number of absolute value less than 1 is bounded between -1 and 1
- The bounds on any number whose absolute value of less than 1 is between -1 and 1

Example 4.2. Translate the following informal statements into equivalent formal statements.

(*i*) "All red cars go fast"

Let $P :=$"is a red car" and let $Q :=$"is fast". Then the above sentence translates to $\forall x$, if $P(x)$ then $Q(x)$.

(*ii*) "Any prime number is not divisible by 4"
Let $P :=$"a prime number" and let $Q :=$"is not divisible by 4". Then the above sentence translates to $\forall x \in \mathbb{Z},$ if $P(x)$ then $Q(x)$. Note that we added the domain $\mathbb{Z}$ since we know that prime numbers are all integers.

As we have seen, there are many different ways of writing the same logical statement. Before we move on, we shall briefly consider some a specific example of how to rewrite a universal conditional statement as a simplle universal statement, a technique which is commonly used in mathematics. Consider the following example.

Example 4.3. "Any prime number is not divisible by 4"
We considered this example before, so if we let $P :=$"is a prime number" and let $Q :=$"is not divisible by 4", then the above sentence translates to $\forall x \in \mathbb{Z},$ if $P(x)$ then $Q(x)$. Notice however, that by a modification of the domain, we can turn this from a universal conditional statement into just a universal statement. Specifically, if we let D be the set of prime numbers, then this statement is equivalent to $\forall x \in D, Q(x)$.

Formalizing, we have the following:

Result 4.4. Suppose "$\forall x \in U,$ if $P(x)$ then $Q(x)$" is some universal conditional statement. Then an equivalent statement is $\forall x \in D, Q(x)$ where D is the subdomain of U with property P.

We consider an example.

Example 4.5. Rewrite the following both as a universal conditional statement and as a universal statement: Any University of Portland student who is a math major is happy.

If we let U be the the of all University of Portland students, let $P :=$ "is a math major" and $Q :=$"is happy". Then a universal conditional statement in formal logic for this sentence is:

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$

or

$\forall$ University of Portland students x, if x is a math major then x is happy

Alternatively, if we let D be the subset of U which consists only of math majors, then a universal statement in formal logic for this sentence is:

$$\forall x \in D, Q(x)$$

or

$$\forall \text{ math majors } x, x \text{ is happy}$$

5. Implicit Quantification

In mathematics, and more generally, we often make universal and existential statements without actually explicitly using the universal or exponential quantifiers. For example, expressions such as:

“An integer is also a real number” (“$\forall$ integers n, n is a real number”)
“A chihuahua is a dog” (“$\forall$ chihuahuas x, x is a dog”)

are universal statements, but the expression “for all” does not appear in either. Likewise, expressions such as:

“Red is the color of some cars” (“$\exists$ a car x, such that x is red”)
“The number 12 is divisible by at least two primes” (“$\exists$ two primes n and m, such that n and m divide 12”)

are existential statements, but the expression “there exists” does not appear in either. When a statement is either universal of existential but the universal or existential quantifiers are not used, we say **implicit universal/existential quantification** has been used. We introduce the following notation for implicit universal quantification (we are not so worried about implicit existential quantification):

Definition 5.1. Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D. The notation $P(x) \implies Q(x)$ means every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or equivalently, $\forall x, P(x) \to Q(x)$. The notation $P(x) \iff Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

We can use this notation to rewrite universal statements without the universal quantifiers.

Example 5.2. Write the expression “All chihuahuas are dogs” without the universal quantifier.

Let $P :=$“is a chihuahua” and $Q :=$“is a dog”. Then this universal statement can be written as

$$P(x) \implies Q(x).$$

Note that it is not the case that

$$P(x) \iff Q(x)$$

since there are dogs which are not chihuahuas.

Homework

(*i*) From the book, pages 86-88: Questions: 1, 5, 7, 10, 14, 16, 17, 18, 19, 22, 23, 26, 31