

Section 2.3: Statements Containing Multiple Quantifiers

In this section, we consider statements such as “there is a person in this company who is in charge of all the paperwork” where more than one quantifier is used referring to two different sets (note that the statement has both an existential quantifier, “there is a person”, and a universal quantifier, “in charge of all the paperwork”). Our first task will be to clarify the interpretation of such statements and then we shall explore further properties of such statements.

1. INTERPRETING STATEMENTS WITH TWO DIFFERENT QUANTIFIERS

In logic and mathematics, it is important that we all interpret a formal statement in the same way, so the first thing we need to do is state exactly how a statement with two different quantifiers should be interpreted. Such statements usually contain multiple variables and are treated in an almost identical way to statement containing a single variable (we are simply substituting in more values). However, since we have not considered such statements in detail beforehand, we shall consider an example to illustrate.

Example 1.1. Let $P(x, y)$ be the sentence “ $x^2 > y$ ”. Are the following true or false.

(i) $P(2, 5)$

We have $P(2, 3) := 4 > 5$ which is clearly false

(ii) $P(3, 8)$

We have $P(3, 8) := 9 > 8$ which is true

Now we need to specify how to interpret statements with multiple quantifiers, or equivalently, determine the truth value of a statement containing two different quantifiers.

Result 1.2. The truth of a statement with two different quantifiers and the negation of such a statement are determined in the in the following ways:

(i) $\forall\exists$ -statement: Consider the statement

$$\forall x \in D \exists y \in E \text{ such that } P(x, y)$$

which reads

“for all x in D , there exists a y in E such that $P(x, y)$ ”

To show that this statement is true, we need to show that given **any** value $x \in D$, we can always find some value $y \in E$ such that $P(x, y)$ is true.

Alternatively, to show this statement is false, we need to show that we can find an $x \in D$ such that for any possible value $y \in E$, $P(x, y)$ is not true, or $\sim P(x, y)$. Thus we have

$$\sim (\forall x \in D \exists y \in E \text{ such that } P(x, y)) \equiv \exists x \in D, \forall y \in E \text{ such that } \sim P(x, y).$$

(ii) $\exists \forall$ -statement: Consider the statement

$$\exists x \in D \forall y \in E \text{ such that } P(x, y)$$

which reads

“there exists x in D , such that for all y in E , $P(x, y)$ ”

This statement will be true if we can find a single value $x \in D$ such that for every $y \in E$, $P(x, y)$ is true.

Alternatively, to show this statement is false, we need to show that for any given $x \in D$, there is a $y \in E$, such that $P(x, y)$ is not true, or $\sim P(x, y)$. Thus we have

$$\sim (\exists x \in D \forall y \in E \text{ such that } P(x, y)) \equiv \forall x \in D \exists y \in E \text{ such that } \sim P(x, y).$$

Remark 1.3. It should be noted that the two different statements above are very similar, but they are saying different things - indeed, a very different process is required to show each are true or false. Therefore, it is important to realize that the order of different quantifiers in a statement with more than one quantifier is extremely important and should not be reversed. In contrast however, if there is a statement with multiple quantifiers which are the same, they can be reordered as we please.

We consider a couple of examples.

Example 1.4. Let $E = D = \{-1, 2, 3, 6, 7\}$. Determine whether the following statements are true.

(i)

$$\forall x \in D \exists y \in E \text{ such that } (x + y) \text{ is even}$$

To show this is true, we need to show that given any x in D , we can find a y in D such that $x + y$ is even. Now if x is odd, we can choose y to be any odd number and the sum will be even. The only other possibility is that $x = 2$, in which case we can choose $y = 2$ and the sum will also be even. Thus, the statement is true.

(ii)

$$\exists x \in D \forall y \in E \text{ such that } (x + y) \text{ is odd}$$

To show this is true, we need to show that there is some x in D , so that for every single y in that, if we take that x and add it to y , the resulting number will be odd. However, this does not appear to be the case i.e. if we choose an odd number, then if we add another odd number, the resulting number will

be even, and if we choose an even number, and add it to an even number, the resulting number would be odd. Therefore, we should try to show it is false by showing its negation holds. However, notice that

$$\begin{aligned} & \sim (\exists x \in D \forall y \in E \text{ such that } (x + y) \text{ is odd}) \\ & \equiv \forall x \in D \exists y \in E \text{ such that } \sim (x + y) \text{ is odd} \\ & \equiv \forall x \in D \exists y \in E \text{ such that } (x + y) \text{ is even} \end{aligned}$$

which is the statement we proved to be true in the previous example. Thus the negation of this statement is true and so the original statement is false.

Remark 1.5. For a statement including multiple quantifiers, we usually leave out the phrase “such that” as illustrated in the next example.

Example 1.6. Determine whether the following statements are true or false:

(i)

$$\forall x \in \mathbb{R} (\exists y \in \mathbb{R} (x < y))$$

To show this statement is true, we need to show that for any real number x , we can always find some other real number y such that $x < y$. If x is an arbitrary real number, let $y = x + 1$. Then y is a real number and $x < y$, and thus the statement is true

(ii)

$$\exists x \in \mathbb{R} (\forall y \in \mathbb{R} (x < y))$$

To show this statement is true, we need to show that for there exists a real number x , such that for every real number y , $x < y$. Note that this is saying that there is some smallest real number, which is abused, so this statement will most likely be false. To show it is false, we need to show for any real number x , there is a real number y such that $x \geq y$, and similar to last time, we can choose $y = x - 1$ and we are done.

Notice that these two statements were almost identical - the only difference was the order of the quantified statements. However, the truth values of these statements were opposite, thus illustrating that the order of different quantified statements is important.

2. TRANSLATING BETWEEN FORMAL AND INFORMAL LANGUAGE

We have seen some basic examples of statements using more than one quantifier and how to work with such statements in logical form. As with previous sections however, the really key to success is being able to translate from informal language to formal language and back again to reveal the truth of such a statement. We finish with a couple of examples of this problem.

Example 2.1. Translate the following sentences into formal logic and then write negations for them and translate the negation back into informal language.

- (i) Every student in this class owns a textbook

Let D be the set students in this class, let E be the set of all text books and let $P(x, y) := “x \text{ owns } y”$. Then this sentence translates to

$$\forall x \in D \exists y \in E P(x, y).$$

The negation of this sentence will be

$$\exists x \in D \forall y \in E \sim P(x, y)$$

which translates as “there is some student in this class who does not own any textbook”.

- (ii) There is a rational number between any two real numbers

In this case there are three different variables. Let $P(x, y, z)$ be the sentence “ x is between z and y ”. Then the statement above can be interpreted into

$$\forall y \in \mathbb{R} \forall z \in \mathbb{R} \exists x \in \mathbb{Q} P(x, y, z)$$

Note that the statement

$$\exists x \in \mathbb{Q} \forall y \in \mathbb{R} \forall z \in \mathbb{R} P(x, y, z)$$

is not the formal interpretation of this sentence since this would say that there is a real number x which lies between every possible pair of real numbers z and y .

Negating this formal statement, we have

$$\exists y \in \mathbb{R} \exists z \in \mathbb{R} \forall x \in \mathbb{Q} \sim P(x, y, z)$$

(using the rules of negating quantifiers) which translates as “There exists two real numbers z and y such that there is no rational number x which lies between z and y ” or removing all variables, “There are two real numbers such that there no rational number lies between them”.

Example 2.2. Rewrite the following statements in informal language and state their negations. Determine whether the statements are true or if their negations are true.

- (i)

$$\exists x \in \mathbb{R} \exists y \in \mathbb{R} (x^2 + y^2 = -1)$$

In informal language, this says “there exists real numbers x and y such that $x^2 + y^2 = -1$ ”. The negation of this statement is “for all real numbers x and y , it is not the case that $x^2 + y^2 = -1$ ” or “for all real numbers x and y , $x^2 + y^2 \neq -1$ ”. Clearly the original statement was false and the negation is true.

(ii)

$$\exists x \in D \forall y \in C P(x, y)$$

where D be the set of people in this class, C be the set of all soft drinks and $P(x, y) := "x$ enjoys drinking $y"$.

This translates as "there exists a person in this class who likes all soft drinks". The negation is "for each person in this room, there is some soft drink they do not like". The truth depends upon the class (of course!).

(iii)

$$\forall y \in C \exists x \in D P(x, y)$$

where D be the set of people in this class, C be the set of all soft drinks and $P(x, y) := "x$ enjoys drinking $y"$.

This translates as "for each soft drink, there is some person who likes it". Note the difference between this and the last statement (for this case, the same person does not have to like all soft drinks, we just know that for each soft drink, at least one person likes it). The negation is "there exists a soft drink which no one in the room likes". The truth depends upon the class (of course!).

Homework

- (i) From the book, pages 108-110: Questions: 1, 4, 10, 11, 13, 15, 18, 19, 21, 28, 32, 33, 40a, 40g, 40c, 43, 53, 54