

Section 2.4: Arguments with Quantified Statements

In this section, we shall generalize the ideas we developed in Section 1.3 to arguments which involve quantified statements. Most of the concepts we shall consider are direct generalizations of ideas from this section, so a thorough understanding of that section will really help out in this section.

1. UNIVERSAL INSTANTIATION

The rule of instantiation is the fundamental of deductive reasoning and it reads as follows:

Result 1.1. If some property is true of everything in a given domain, then it is true of any particular thing from that domain. In formal language, this says

$$\begin{array}{l} \forall x \in D, P(x) \\ a \in D \\ \circ\circ P(a) \end{array}$$

Most of the arguments we shall build up in deductive reasoning will hinge upon this rule. The following are classic examples of applications of the rule of instantiation.

Example 1.2. All of the following are examples of the rule of instantiation:

(i)

All men are mortal
Socrates is a man
Therefore, Socrates is mortal

(ii)

All integers are rational numbers
2 is an integer
Therefore, 2 is a rational number

We shall now consider how the arguments we developed can be modified to quantified statements through the use of the rule of instantiation. For each argument, we shall state both the formal and the informal versions.

2. UNIVERSAL MODUS PONENS AND MODUS TOLLENS

The first arguments we shall generalize are modus ponens and modus tollens.

Result 2.1. (Universal Modus Ponens)

Formal Version	Informal Version
$\forall x, P(x) \rightarrow Q(x)$	For all x , if x makes $P(x)$ true, then it makes $Q(x)$ true
$P(a)$	a makes $P(x)$ true
$\circ\circ Q(a)$	Therefore, a makes $Q(x)$ true

Result 2.2. (Universal Modus Tollens)

Formal Version	Informal Version
$\forall x, P(x) \rightarrow Q(x)$	For all x , if x makes $P(x)$ true, then it makes $Q(x)$ true
$\sim Q(a)$	a makes $P(x)$ false
$\circ\circ \sim P(a)$	Therefore, a makes $P(x)$ false

We consider a couple of examples of how to recognize and use modus ponens and modus tollens.

Example 2.3. Rewrite the following argument in formal logic and determine if it is valid.

All people who work hard in this class will get a good grade in this class

I worked hard in this class

Therefore, I will get a good grade in this class.

Let $P(x) :=$ “ x worked hard in this class”, let $Q(x) :=$ “ x will get a good grade in this class” and let D be the set of all people in this class. Then this argument translates to:

$$\begin{aligned} \forall x, P(x) \rightarrow Q(x) \\ P(\text{me}) \\ \circ\circ Q(\text{me}) \end{aligned}$$

This argument is universal modus ponens and is therefore a valid argument.

Example 2.4. Determine the conclusion which can be inferred from the following argument:

If n is an even integer, then n^2 is an even integer

$3^2 = 9$ is not an even integer

This has the argument form of modus tollens. Specifically, let $P(x) :=$ “ x is an even integer”, $Q(x) :=$ “ x^2 is an even integer”. Then the in formal logic, the statements above become:

$$\begin{aligned}\forall x, P(x) \rightarrow Q(x) \\ \sim Q(3)\end{aligned}$$

Using modus tollens, the conclusion of this argument is $\sim P(3)$, or “3 is not an even integer”.

We shall now use formal logic to prove a mathematical fact. We shall include most details to illustrate the power of the arguments we have discussed.

Example 2.5. Use logic to prove that $\forall n \in \mathbb{Z}, n \text{ is even} \rightarrow n^2 \text{ is even}$.

To prove this conditional statement, we need to show that any n which satisfies “ n is even” also satisfies “ n^2 is even”.

(i) Assume n is an even integer. The definition of an even integer is: “ $\forall x \in \mathbb{Z}, x \text{ is even} \leftrightarrow \exists y \in \mathbb{Z}, x = 2y$ ”. Since n is an even integer, $\circ\circ$, $n = 2m$ for some integer m (modus tollens). Note that the definition of something is a biconditional statement!

(ii) From the rules of arithmetic, we know

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall a \in \mathbb{R}^+, (xy)^a = x^a y^a$$

m , 2, and 2 are all particular real numbers, so $(2m)^2 = 2^2 m^2$ using universal instantiation.

(iii) From the rules of arithmetic, we know

$$\forall x \in \mathbb{R}, x^2 = x * x$$

2 is a particular real number, so $2^2 = 2 * 2$ using universal instantiation.

(iv) We conclude that $n^2 = 2 * (2m^2)$. However, we know that “ $\forall x \in \mathbb{Z}, x \text{ is even} \leftrightarrow \exists y \in \mathbb{Z}, x = 2y$ ”. Since this is a biconditional, it can be read in both directions. In particular, since $y = 2m^2$ is an integer and $n = 2 * y$, it follows that n^2 is an even integer.

Note that there we actually excluded a number of steps in this argument (because they were similar to the ones we already stated coming from the elementary rules of arithmetic). In order to make a completely formal argument, we should really have justified every single step, but the idea of this example was to give a general gist of how the arguments we have considered can be used in mathematical proof.

3. TESTING THE VALIDITY OF AN ARGUMENT

We take the definition of an argument the same as our original definition - it is a list of statements, all except the last we call premises, and

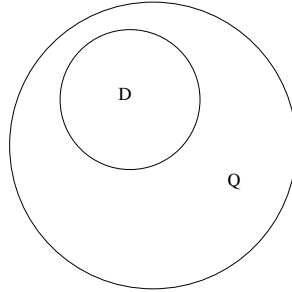
the last we call the conclusion. Likewise, we give a similar definition for the validity of an argument. Specifically, we define validity as follows:

Definition 3.1. To say that an argument form is valid means the following: No matter what predicates are substituted for the predicate symbols in the premises, if the resulting premise statements are true, then the conclusion is also true. Alternatively, an argument is valid if and only if whenever all of the premises are true, the conclusion is true

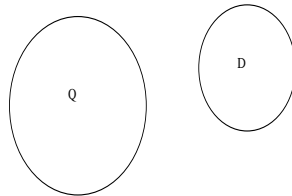
In general, showing that an argument is valid is a fairly difficult thing to do (as we have seen in previous sections). In arguments with quantifiers, one technique you can use to try to determine whether an argument is invalid is through the use of diagrams which shows how the objects in the argument are related. For the statement

$$\forall x, x \in D \rightarrow Q(x)$$

means that if x is in D , then $Q(x)$ must be true. Specifically, this means that the set of all things in D will be a subset of all the things for which $Q(x)$ is true. To represent this, we draw two circles, one inside the other. In the inner circle, we write D and on the outer circle, we write Q as illustrated below.



This then tells us that everything in D must be in Q since the circle of all things in D is contained in Q . Notice however, that it does not tell us whether something in Q is in D (indeed it would be wrong to assume so). In a similar theme, if we have an expression such as $\forall x, x \in D \rightarrow \sim Q(x)$, we would draw two circles representing the domain of D and all the things for which Q is true, and in this case, the two circles would be drawn apart from each other.



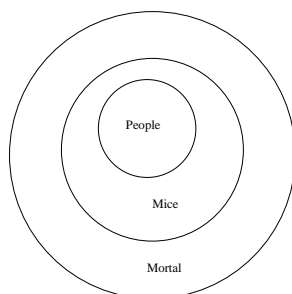
Finally, if a given relation between two properties is not given, or we know that some of one domain share properties with some of another domain, then we may need to draw circles which intersect. To determine the validity of an argument, we draw all possible diagrams which satisfy the truth values of the premisses, and then consider whether any

of these diagrams violate the conclusion. We illustrate with a couple of examples.

Example 3.2. Use diagrams to determine if the following arguments are valid or invalid.

- (i) All people are mice
All mice are mortal
Therefore, all people are mortal.

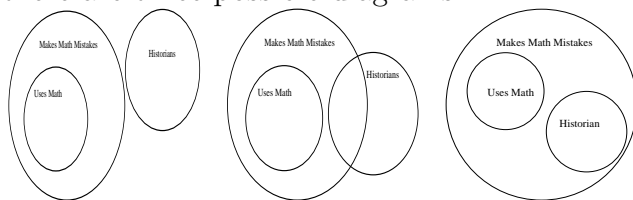
Since all people are mice, we draw a circle representing people inside a circle representing mice. Next, since all mice are mortal, we draw a third circle representing mortals around the circle containing mice. Note that the only possible diagram is the following:



Since the circle containing people is in the circle containing mortals, it follows that all people are mortals, and thus the argument is valid.

- (ii) Every person who uses math in their jobs makes mathematical errors
No historian uses math
Therefore, no historians make mathematical errors

Since every person who uses math in their jobs makes mathematical mistakes, we draw a circle representing people who use math in their job inside the circle of people who make mathematical mistakes. Since a historian does not use math, the circle representing all historians does not intersect the circle of people who use math. However, nothing is said about the intersection of the people who make mathematical mistakes, so there are three possible diagrams:



In the first case, the conclusion is true. However, in the second and third diagrams, the conclusion is not necessarily true.

Thus this argument is not valid since the truth of the conclusion does not follow from the truth of the premises.

Warning. When using diagrams to check for validity, make sure you consider **all** possible diagrams, else your proof may not be valid.

4. INVERSE AND CONVERSE ERRORS

For the last example we considered, we saw an invalid argument. As we mentioned in Section 1.3, there are many different types of invalid arguments (far too numerous to mention). However, there are a couple of invalid arguments which are often considered valid by mistake - these are the quantified forms of the converse and inverse errors. We summarize.

Result 4.1. (Converse Error)

Formal Version	Informal Version
$\forall x, P(x) \rightarrow Q(x)$	For all x , if x makes $P(x)$ true, then it makes $Q(x)$ true
$Q(a)$	a makes $Q(x)$ true
$\circ\circ P(a)$	Therefore, a makes $P(x)$ true

Result 4.2. (Inverse Error)

Formal Version	Informal Version
$\forall x, P(x) \rightarrow Q(x)$	For all x , if x makes $P(x)$ true, then it makes $Q(x)$ true
$\sim P(a)$	a makes $Q(x)$ false
$\circ\circ \sim Q(a)$	Therefore, a makes $Q(x)$ false

Invalid arguments like the above are often used in advertising, politics, law etc. to endorse products or justify examples. We illustrate with some examples.

Example 4.3. Determine the validity of the following arguments.

- All the people who are in this class are good at math
- (i) Fred is good at math
Therefore, Fred is in his class

This is a case of the converse error - just because Fred is good at math does not mean he is in the class (it is the other way around).

- Anyone who thinks raising taxes helps the economy doesn't understand economics
- (ii) I understand economics
Therefore, I don't think that raising taxes will help the economy

This is a valid argument (modus tollens).

- Anyone who was not present in the room could not have committed the crime
- (iii) The defendant was present in the room
Therefore, you must have committed the crime

This is a case of the inverse error - just because the defendant was present does not mean the defendant committed the crime.

5. CREATING ADDITIONAL FORMS OF ARGUMENT

Just as we generalized modus ponens and modus tollens to quantified statements, we can also do it with many of the other argument forms we introduced in Section 1.3 (see the text for examples).

Homework

- (i) From the book, pages 122-124: Questions: 2, 3, 5, 11, 13, 14, 18, 19, 22, 26, 31, 33