

## Section 3.2: Direct Proof and Counterexample 2

In the last section, we considered properties of integers and proved and disproved many different facts about integers. In this section, we shall consider rational numbers. First, we shall formally define a rational number, then we shall prove some elementary facts about rational numbers.

### 1. THE DEFINITION OF A RATIONAL NUMBER

We start with the formal definition of a rational number.

**Definition 1.1.** A real number  $r$  is rational if and only if it can be expressed as a quotient of two integers with a non-zero denominator. In notation,

$$r \text{ is rational} \iff \exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}^* \text{ such that } r = \frac{a}{b}$$

A number which is not rational is called irrational.

**Example 1.2.** Which of the following are rational numbers and which are irrational numbers:

$$0/5, 0.1, 3, 2/0, 0.\bar{1}2$$

- (i)  $0/5$  is a rational number since it is the quotient of two integers, the bottom which is not zero.
- (ii)  $0.1$  is a rational number since it is equal to  $1/10$  which is the quotient of two integers, the bottom which is not zero.
- (iii)  $3$  is a rational number since it is equal to  $3/1$  which is the quotient of two integers, the bottom which is not zero.
- (iv)  $2/0$  is not a rational number since it is equal to the quotient of two integers, the bottom which is zero. This usually implies that it is rational. However, in this case it doesn't because it is not even a number i.e. it is undefined.
- (v)  $0.\bar{1}2$  is a rational number. Specifically, let  $x = 0.\bar{1}2$ . Then

$$100x - x = 12 \text{ so } 99x = 12 \text{ and hence } x = 12/99 = 4/33$$

so since  $0.\bar{1}2$  is equal to  $4/33$  which is the quotient of two integers, the bottom which is not zero, it is a rational number (in fact any repeated decimal is rational).

### 2. ELEMENTARY FACTS ABOUT RATIONAL NUMBERS

Just as with integers, there are many different statements we can prove or disprove about rational numbers. We illustrate with some important examples.

**Theorem 2.1.** *Every integer is a rational number.*

*Proof.* Suppose that  $n$  is an integer. Then  $n = n/1$  using the elementary rules of arithmetic. It follows that  $n$  is expressible in the form  $a/b$  where  $a$  and  $b$  are integers and  $b \neq 0$ . Thus by definition,  $n$  is rational.  $\square$

**Theorem 2.2.** *The sum of two rational numbers is itself rational.*

*Proof.* Suppose that  $r$  and  $q$  are rational numbers. Then there exist integers  $a, b, c, d$  with  $b \neq 0$  and  $d \neq 0$  such that  $r = a/b$  and  $q = c/d$ . Then we have

$$r + q = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Note that since  $a, b, c, d$  are all integers, so is  $ad + bc$  and  $bd$ . Also, since  $b$  and  $d$  are nonzero, so is  $bd$ . Thus  $r + q$  can be written as the quotient of two integers, the bottom of which is nonzero and hence it is rational.  $\square$

**Example 2.3.** Prove or disprove the following statement: “If  $a$  is an odd integer, then  $a^2 + a$  is even”

It seems that this should be true since the square of an odd integer is odd, and the sum of two odd integers is even, so we shall try to prove it.

Suppose that  $a$  is an odd integer. Then there exists an integer  $k$  such that  $a = 2k + 1$ . Then

$$\begin{aligned} a^2 + a &= (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 \\ &= 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1). \end{aligned}$$

Since  $k$  is an integer, so

**Example 2.4.** A corollary is a statement which is a consequence of a theorem (or can be directly derived from a theorem). Prove the following corollary to Theorems 2.1 and 2.2.

**Corollary 2.5.** *The sum of any two integers is a rational number.*

*Proof.* Suppose that  $a$  and  $b$  are integers. Then by Theorem 2.1, both  $a$  and  $b$  are rational. However, according to Theorem 2.2, the sum of two rational numbers is rational. Thus  $a + b$  is rational.  $\square$

### 3. DERIVING NEW MATHEMATICS FROM OLD

Very rarely in mathematics does a proof go back to the real basics - usually results are derived from previous known results. Elementary number theory of integers and rational numbers is a well established field, so there are many already known results which can be used to help

prove certain statements. As a general case, unless we specifically state that we want to **prove something directly from the definitions**, we shall allow certain established facts to be incorporated in our proofs. Typical known results are the following:

- (i) The sum, product and difference of even integers are even
- (ii) The sum and difference of two odd integers is even but the product of two odd integers is odd
- (iii) The product of an even and odd integer is even
- (iv) The sum of an odd and even integer is odd
- (v) The difference of an odd and even integer is odd
- (vi) The difference of an even and odd integer is odd

We illustrate how previous results can be used to prove further results.

**Example 3.1.** Prove the statement: “If  $a$  is an odd integer, then  $a^2 + a$  is even” (we did this above)

Suppose that  $a$  is odd. Then  $a^2 = a \cdot a$ , the product of two odd integers, so is odd. Similarly,  $a^2 + a$  will be the sum of two odd integers, so will be even. This  $a^2 + a$  will be even.

### Homework

- (i) From the book, pages 139-141: Questions: 2, 3, 7, 8, 10, 13, 20, 26, 27, 31, 32, 34, 35