Section 4.1: Sequences and Series

In this section, we shall introduce the idea of sequences and series as a necessary tool to develop the proof technique called mathematical induction. Most of the material in this section should be review from Calculus 2.

1. Sequences

We start with a formal definition of a sequence with some related terminology and give some easy examples.

Definition 1.1. A sequence is a succession of numbers

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

written in a specificied order. We call a_m the initial term, and in general, a term a_k in a sequence is called the kth term where k is called the subscript or index of the term. If the sequence terminates at a_n, then we call the term a_n the final term. A **general formula** for a sequence is an expression for the values of a_k in terms of k.

Remark 1.2. A sequence does not have to start at $m = 1$.

Remark 1.3. Note that sequences can be defined recursively i.e. in terms of the previous terms.

The following are some simple examples of sequences given by general formulas.

Example 1.4. (i) We define a sequence as $a_k = k$. This sequence will consist of the integers written in consecutive order $1, 2, 3, 4, \ldots$.

(ii) We can define an interesting sequence as follows,

$a_k = k$th decimal place in the decimal expansion of π.

(iii) We can define the recursive sequence $a_1 = 1$, $a_2 = 1$ and for $n > 2$ we define $a_n = a_{n-1} + a_{n-2}$. The first few terms are: $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$. This sequence is called the Fibonacci sequence and arises in many strange natural and physical situations.

(iv) Consider the sequence

$$a_k = (-1)^k k!$$

Note that consecutive terms changes sign. We call such a sequence an **alternating sequence**.

We have seen a few examples of sequences given a general formula for their $k$the terms. Another useful skill is being able to determine a general formula given a few terms of the sequence.

Example 1.5. Find formulas for the following sequences (either recursive or as a function of n).

(*i*)

$$\{1, 4, 9, 16, 25, 36, \dots\}.$$

Observe that this sequence is the ascending list of squares of integers. Therefore, a formula will be $a_k = k^2$.

(*ii*)

$$\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots\}.$$

In this case, the sequence is changing from positive to negative and the denominator is ascending power of 2. Therefore a formula will be

$$a_k = (-1)^{k+1}\frac{1}{2^k}.$$

(*iii*)

$$\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \dots\}.$$

The numerator is running over consecutive odd numbers and the denominator is running over consecutive even numbers. Thus we have

$$a_k = \frac{2k-1}{2k}.$$

(*iv*)

$$\{1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1 \dots\}.$$

This function does not look like it can be described nicely as a function of n. Therefore, we try to define it recursively. Notice that,

$$a_1 = 1$$
$$a_2 = 0$$
$$a_3 = a_2 - a_1 = -1$$
$$a_4 = a_3 - a_2 = -1 - 0 = -1$$
$$a_5 = a_4 - a_3 = -1 - (-1) = 0$$
$$a_6 = a_5 - a_4 = 0 - (-1) = 1$$
$$a_7 = a_6 - a_5 = 1 - 0 = 1$$
$$a_8 = a_7 - a_6 = 1 - 1 = 0$$
$$a_9 = a_8 - a_7 = 0 - 1 = -1$$
$$a_{10} = a_9 - a_8 = -1 - 0 = -1$$
$$a_{11} = -1 - (-1) = 0$$

so the sequence will be $a_1 = 0$, $a_2 = 1$ and $a_k = a_{k-1} - a_{k-2}$ for $n > 2$.

2. Series

Recall the following definition of a series.

Definition 2.1. If $a_m, a_{m+1}, \ldots$ is a sequence, we can add up all the terms $a_m + a_{m+1} + \ldots$. We call this sum a series and the expression $a_m + a_{m+1} + \ldots$ the expanded form of the series. If the final term in the sequence is a_n, then we denote it by

$$\sum_{i=m}^{n} a_i$$

(called the summation form of the series) and call m the lower limit and n the upper limit. If the sequence is infinite, we replace the upper limit by ∞.

We consider some examples of how to work with series.

Example 2.2. Evaluate the value of the series

$$\sum_{i=3}^{6} \frac{(-1)^i}{i^2}$$

In expanded form, we have

$$\sum_{i=3}^{6} \frac{(-1)^i}{i^2} = -\frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} = \frac{73}{1200}.$$

Example 2.3. Write the series

$$\sum_{i=5}^{n+1} \frac{i+1}{i^2+1}$$

in expanded form

In expanded form, we have

$$\sum_{i=5}^{n+1} \frac{i+1}{i^2+1} = \frac{6}{26} + \frac{7}{37} + \cdots + \frac{n+2}{(n+1)^2+1}.$$

For a finite series, the final value of a series is often related to the upper limit and under such circumstances, we have to be careful when calculating values of such sums. We illustrate.

Example 2.4. Write the series

$$n + \frac{n-1}{2} + \frac{n-2}{4} + \cdots + \frac{1}{2^{n-1}}$$

in summation form and find the values for $n = 1$, $n = 2$ and $n = 3$.

The general term for this series will be

$$a_k = \frac{n-k}{2^k}$$

for $0 \leqslant k \leqslant (n-1)$. Therefore, in summation notation, we have

$$n + \frac{n-1}{2} + \frac{n-2}{4} + \cdots + \frac{1}{2^{n-1}} = \sum_{k=0}^{n-1} \frac{n-k}{2^k}.$$

To find the different values of the sum, we need to return to expanded form. Specifically, we have

$$\text{when } n = 1 \text{ the sum is: } 1$$

$$\text{when } n = 2 \text{ the sum is: } 2 + \frac{1}{2}$$

$$\text{when } n = 3 \text{ the sum is: } 3 + \frac{2}{2} + \frac{1}{4}$$

3. Products

Just as we can sum the terms in a sequence, we can determine the product of terms in a sequence.

Definition 3.1. We define the product of the sequence a_k from m to n to be

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdots a_n.$$

Calculations of products is similar to that of sums and only requires simple arithmetic after a product has been written in expanded form, so we shall not do any specific examples. However, there is on particular type of product which will be very important in later sections.

Definition 3.2. For each positive integer n, the quantity n factorial, denoted $n!$ is the product of all integers from 1 to n i.e.

$$n! = \prod_{i=1}^{n} i.$$

We define $0! = 1$.

Example 3.3. Compute the following:

(i)

$$\frac{4!}{0!}$$

We have

$$\frac{4!}{0!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1} = 24$$

(ii)

$$\frac{n!}{(n-k)!}$$

We have

$$\frac{n!}{(n-k)!} = \frac{n\cdot(n-1)\cdots(n-k)\cdot(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = n\cdot(n-1)\cdots(n-k+1).$$

As is probably expected, there are certain rules from elementary arithmetic which generalize to sums and products. Specifically, we have the following.

Theorem 3.4. *Suppose that $a_m, a_{m+1}, \ldots$ and $b_m, b_{m+1}, \ldots$ are sequences and c is some real number. Then the following are true:*

(*i*)

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$$

(*ii*)

$$c\cdot\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c\cdot a_k$$

(*iii*)

$$\prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k = \prod_{k=m}^{n}(a_k\cdot b_k)$$

Such formulas can be used to combine multiple sums and products.

4. Changing Variables

Often it may be useful, or required, to change the variable in a given series or product. Under such circumstances, we need to be careful to make sure we also change the limits as well as the expression in the product or sum. We shall illustrate with an example of how to perform such an action.

Example 4.1. Combine the summations

$$\sum_{k=1}^{n}(2k-3) + 2\cdot\sum_{k=2}^{n+1}(2k^2+1)$$

to make a single summation

First we have

$$\sum_{k=1}^{n}(2k-3) + 2\cdot\sum_{k=2}^{n+1}(2k^2+1) = \sum_{k=1}^{n}(2k-3) + \sum_{k=2}^{n+1}(4k^2+2)$$

We would like to combine these sums, but we cannot use the law since the indexes are different. Therefore, we shall make the change of variable $k = j+1$ in the second sum. First, changing the limits, if $k = 2$, then $j+1=1$, so $j=1$, and similarly, if $k=n+1$, then $j+1=n+1$ or $j=n$. Therefore, we have

$$\sum_{k=2}^{n+1}(4k^2+2)=\sum_{j=1}^{n}(4(j+1)^2+2).$$

Since changing the variable j to k does not change the value of the sum, we have

$$\sum_{j=1}^{n}(4(j+1)^2+2)=\sum_{k=1}^{n}(4(k+1)^2+2).$$

Thus

$$\sum_{k=1}^{n}(2k-3)+2\cdot\sum_{k=2}^{n+1}(2k^2+1)=\sum_{k=1}^{n}(2k-3)+\sum_{k=2}^{n+1}(4k^2+2)=\sum_{k=1}^{n}(2k-3)+4(j+1)^2+2).$$

Homework

(i) From the book, pages 213-215: Questions: 2, 6, 12, 16, 18a, 18c, 19, 28, 31, 33, 35, 41, 43, 46, 48, 52, 54, 59, 60,