

Section 4.2: Mathematical Induction 1

Over the next couple of sections, we shall consider a method of proof called mathematical induction. Induction is fairly complicated, but a very useful proof technique, specifically when we are trying to prove properties relating integers, or more generally, properties relating sets which can be put in 1 – 1 correspondence with the integers. In this section, we shall introduce the idea of mathematical induction, discuss why it is a valid method of proof and consider the general approach of how to prove something using induction. We shall also consider some classic examples of proofs by induction.

1. THE PRINCIPLE OF MATHEMATICAL INDUCTION

We start with a statement of the principle of mathematical induction:

Result 1.1. Let $P(n)$ be a property that is defined for integers n and let a be a fixed integer. Suppose the following two statements are true:

- (i) $P(a)$ is true
- (ii) For all integers $k \geq a$, if $P(k)$ is true, then $P(k + 1)$ is true.

Then the statement

$$\forall n \geq a, P(n)$$

is true.

The obvious question to ask is: “why induction is a valid form of proof?” The basic idea is follows:

- (i) Suppose the statements in the principle of induction are true. We want to show that this means $P(n)$ will be true for any $n \geq a$.
- (ii) First note that since $n \geq a$, there is a positive integer k such that $n = a + k$.
- (iii) Now, since $P(a)$ is true (by assumption), and since for all k , if $P(k)$ is true, then so is $P(k + 1)$, it follows that $P(a + 1)$ is true.
- (iv) In a similar theme, we know that $P(a + 2)$ will be true. Likewise, $P(a + 3)$ will be true. Continuing this process, we finally get that $P(a + k) = P(n)$ will be true.

The next question is “how can we prove a mathematical statement using induction?” First, we need to recognize when induction seems a reasonable proof technique. We should consider induction when we are considering the truth value of statements of the form: “For all integers $n \geq a$, property $P(n)$ is true”. When given such a statement, to prove it using induction, take the following steps:

Step 1. (*Base Step*) Show the property is true for $P(a)$

Step 2. (*Induction Step*) Show that for all integers $k \geq a$, if we suppose that $P(k)$ is true, then $P(k+1)$ is true. In order to do this, we usually do the following:

- (i) (*Inductive hypothesis*) Suppose that $P(k)$ is true for some arbitrarily chosen $k \geq a$. Write out what this means.
- (ii) Show P is true for $k+1$ only under the assumption that $P(k)$ is true. To do this, it may be useful to write out what it would mean for $P(k+1)$ to be true.

2. EXAMPLES OF INDUCTION

The only way to really learn induction is to examine in detail a number of examples. We start with a simple example.

Example 2.1. Show that any amount of money of at least 14 cents can be made up from 3 cent and 8 cent coins.

The first thing we need to do is give a formal statement. We are saying that any number $k \geq 14$ can be written as a sum of a multiple of 3 and a multiple of 5. Therefore, the statement we are trying to prove is $P(n) =$ "There exists x and y such that $n = 3x + 8y$ ". This is a statement about every integer $n \geq 14$, so we can attempt to use induction. Therefore, we shall state this as a theorem and attempt to prove it.

Theorem 2.2. *For any integer $n \geq 14$, there exists integers x and y such that $n = 3x + 8y$.*

Proof. First we need to prove the base case. Notice that if $y = 1$ and $x = 2$, then $14 = 3 \cdot 2 + 1 \cdot 8$, so the base statement $P(14)$ holds.

Now assume that the statement holds for an arbitrary but fixed integer k i.e. we are assuming there exists integers x and y such that $k = 3x + 8y$. We want to show that $P(k+1)$ is true i.e. that there exists integers a and b such that $k+1 = 3a + 8b$.

We know

$$k + 1 = 3x + 8y + 1$$

for x and y given above since we are assuming $k = 3x + 8y$. If $y \neq 0$, then

$$k+1 = 3x+8y+1 = 3x+8(y-1)+8+1 = 3x+8(y-1)+9 = 3(x+3)+8(y-1).$$

In particular, $k+1$ can be written in the form $3a + 8b$ where $a = x + 3$ and $b = y - 1$. If $y = 0$, then $x \geq 5$ (since we are assuming that $k \geq 14$). Therefore

$$k + 1 = 3x + 1 = 3(x - 5) + 15 + 1 = 3(x - 5) + 8 \cdot 2.$$

In particular, $k + 1$ can be written in the form $3a + 8b$ where $a = x - 5$ and $b = 2$. Since this gives all possible choices for y , the proof follows by induction. \square

We shall now prove a couple of classical mathematical results whose proofs require the principle of induction.

Theorem 2.3. *For all integers $n \geq 1$,*

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Proof. Here we are considering the statement $P(n) :=$ “For any $n \geq 1$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ ”. We shall use induction.

(Base Case) When $n = 1$, we are simply asking for the sum of the first integer which will be 1. Using the formula with $n = 1$, we have

$$\frac{1 \cdot (1 + 1)}{2} = 1.$$

So $P(1)$ holds.

(Induction Step) Assume that the result holds for k i.e.

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}.$$

We want to show that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Consider the sum

$$\sum_{i=1}^{k+1} i = (k+1) + \sum_{i=1}^k i.$$

By the induction hypothesis, we have

$$\begin{aligned} \sum_{i=1}^{k+1} i &= (k+1) + \sum_{i=1}^k i = (k+1) + \frac{k(k+1)}{2} = \frac{2k+2}{2} + \frac{k(k+1)}{2} \\ &= \frac{2k+2+k^2+k}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

which is exactly what we were trying to show. \square

Example 2.4. Calculate the following:

$$(i) 8 + 9 + 10 + \cdots + 38$$

We have

$$8 + 9 + 10 + \cdots + 38 = \sum_{i=1}^{38} i - \sum_{i=1}^7 i = \frac{38 \cdot 39}{2} - \frac{7 \cdot 8}{2} = 741 - 28 = 713.$$

$$(ii) s + (s + 1) + (s + 2) + \cdots + (s + r)$$

We have

$$\begin{aligned} s + (s + 1) + \cdots + (s + r) &= \sum_{i=1}^{s+r} i - \sum_{i=1}^{s-1} i \\ &= \frac{(s+r)(s+r+1)}{2} - \frac{s(s+1)}{2} = \frac{(s+r)(s+r+1) - s(s+1)}{2}. \end{aligned}$$

Theorem 2.5. For any real number r except 1 and any integer $n \geq 0$,

$$\sum_{i=0}^n r^i = \frac{r^{n+1} - 1}{r - 1}$$

Proof. Here we are considering the statement $P(n) :=$ “For any real number $r \neq 1$ and for any integer $n \geq 0$, $\sum_{i=0}^n r^i = \frac{r^{n+1}-1}{r-1}$ ”. We shall use induction.

(Base Case) When $n = 0$, we are simply asking for the value of r^0 . For any real number r ,

$$r^0 = 1 = \frac{r^1 - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$$

So $P(0)$ holds.

(Induction Step) Assume that the result holds for k i.e.

$$\sum_{i=0}^k r^i = \frac{r^{k+1} - 1}{r - 1}.$$

We want to show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}.$$

Consider the sum

$$\sum_{i=0}^{k+1} r^i = r^{k+1} + \sum_{i=0}^k r^i.$$

By the induction hypothesis, we have

$$\begin{aligned} \sum_{i=0}^{k+1} r^i &= r^{k+1} + \sum_{i=0}^k r^i = r^{k+1} + \frac{r^{k+1} - 1}{r - 1} = \frac{r^{k+2} - r^{k+1}}{r - 1} + \frac{r^{k+1} - 1}{r - 1} \\ &= \frac{r^{k+2} - r^{k+1} + r^{k+1} - 1}{r - 1} = \frac{r^{k+2} - 1}{r - 1} \end{aligned}$$

which is exactly what we were trying to show.

□

Homework

- (i) From the book, pages 226-227: Questions: 2, 3, 6, 10, 14, 16, 20, 26, 27, 30, 32