Section 4.3: Mathematical Induction 2

In the last section we introduced the principle of mathematical induction and considered some classical examples of mathematical proofs by induction. In this section we shall continue this theme and consider some further proofs using induction.

1. Recognizing when to use Induction

There are two big hurdles to succeeding in proving results using induction, though arguably the second is a consequence of the first. These are:

- (i) Recognizing when to use mathematical induction
- (ii) Formulating the statement of what is to be proved using induction

As a general case, induction is used to prove a statement regarding an observed pattern which arises in mathematics. For example, when the formula for the first n integers arose, it was highly unlikely that the person who derived the formula just thought it up without any further reference. It is much more likely that they observed the pattern through explicit calculation, then conjectured the result and use mathematical induction to prove it. Therefore, the most likely candidate for a statement which could possibly be proved using induction is a statement regarding the recurrence of some pattern.
The problem of formulating the induction really comes down to understanding the statement to be proved. If we have a clear statement of exactly what we are trying to prove, then we are likely to be able to formulate exactly what we need to prove inductively. We illustrate with an example.

Example 1.1. Observe that

$$\frac{1}{1\cdot 3}=\frac{1}{3},\qquad \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}=\frac{2}{5},\qquad \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}=\frac{3}{7}$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}=\frac{4}{9}.$$

Guess a general formula for this pattern and the prove the formula holds using induction.

First we guess that the formula is

$$\sum_{i=1}^{n}\frac{1}{(2i-1)(2i+1)}=\frac{n}{2n+1}.$$

We shall now attempt to prove this formula using induction. For $n = 1$, we have

$$\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1 \cdot 3} = \frac{1}{3} = \frac{1}{2 \cdot 1 + 1},$$

so the base case holds. We now assume that the result holds for $n = k$ and prove it for $n = k + 1$ i.e. we need to show that

$$\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{(k+1)}{(2(k+1)+1)}.$$

We have

$$\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{(2(k+1)-1)(2(k+1)+1)} + \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)}$$
$$= \frac{1}{(2k+1)(2k+3)} + \frac{k}{2k+1}$$

by the induction hypothesis and simple algebraic manipulation. But then,

$$\frac{1}{(2k+1)(2k+3)} + \frac{k}{2k+1} = \frac{1}{(2k+1)(2k+3)} + \frac{(2k+3)k}{(2k+3)(2k+1)}$$
$$= \frac{1+2k^2+3k}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{(2(k+1)+1)}.$$

2. Further Examples using Induction

We shall consider a number of different types of statements which can be proved using induction.

Proposition 2.1. *$n^3 - n$ is divisible by 6 for every integer $n \geqslant 2$.*

Proof. Base Case: When $n = 2$, we have $2^3 - 2 = 6$ which is divisible by 6.
Induction step: Assume that $k^3 - k$ is divisible by 6. We want to show that $(k+1)^3 - (k+1)$ is divisible by 6. Using simple algebra, we have

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k.$$

Rewriting, we have

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + 3k = (k^3 - k) - 3k(k-1).$$

We know that 6 divides $k^3 - k$ by the induction hypothesis. Clearly 3 divides $3k(k-1)$, and notice that 2 must also divide $3k(k-1)$ since $k \geqslant 2$, and hence either k or $k - 1$ will be even. Therefore, 6 divides $3k(k-1)$ and $k^3 - k$ and hence must divide

$$(k+1)^3 - (k+1) = (k^3 - k) - 3k(k-1)$$

proving the result.

□

Proposition 2.2. $2^n < (n+2)!$ *for all integers* $n \geqslant 0$

Proof. Base Case: $n = 0$, we have $2^0 = 1 < (0+2)! = 2$
Induction Step: Assume the result holds for k and show that it holds for $k+1$ i.e. we want to show that $2^k < ((k+1)+2)!$. We have

$$((k+1)+2)! = (k+3)! = (k+3) \cdot (k+2)!$$

Using the fact that $2^k < (k+2)!$, we have

$$(k+3) \cdot (k+2)! > 2^k(k+3) > 2^{k+1}$$

since $k+3 \geqslant 2$.

□

We finish with an example proving the general nth term of a sequence using induction.

Example 2.3. Show that the sequence $a_1, a_2, \ldots$ defined recursively as $a_k = 7a_{k-1}$ and $a_1 = 3$ has general nth term $a_n = 3 \cdot 7^{n-1}$ for $n \geqslant 1$.

Base Case: $n = 1$, we have $a_1 = 2$ and using the formula, $a_1 = 3 \cdot 7^0 = 3$, so the base case holds.
Induction Step: Assume that formula holds for k and show that it holds for $k+1$ i.e. show that $a_{k+1} = 3 \cdot 7^k$. We have

$$a_{k+1} = 7 \cdot a_k = 7 \cdot 3 \cdot 7^{k-1} = 3 \cdot 7^k$$

using the induction hypothesis. Hence the formula holds.

Homework

(*i*) From the book, pages 233-235: Questions: 2, 6, 10, 15, 17, 21, 26, 28, 30, 31