Section 5.1: Basic Definitions of Set Theory

Many people believe the topic of set theory lies at the very foundations of mathematics - indeed, all mathematics, from simple arithmetic to advanced calculus, can be derived from set theory. In this chapter, we shall take a brief look at the ideas behind set theory. Specifically, we shall consider the basic definitions and properties of sets as well as proofs of important theorems about sets and counterexamples to common conjectures about sets. Before we consider this however, we need to start by building up a dictionary of the basics in set theory.

1. Basic Set Theory Notation

The terms "set" and "element" are usually considered primitive terms meaning they cannot be defined (the reason we cannot define them is that in order to define them, we would have to assume they exist, hence begging the question). Naïvely speaking, a set is a collection of objects called elements. A set is completely determined by its elements, where repetition and ordering of elements is irrelevant (this is usually called the axiom of extension). The following notation for sets is standard as we stated before:

Notation 1.1. Suppose that S is some set

(i) If x is an element of S, then we write $x \in S$

(ii) If x is not an element of S, we write $x \notin S$

(iii) When we want to see the elements of a set, we can write them as a list inside curly braces:

$$S := \{a, b, c\} \text{ or when } S \text{ is infinite } S = \{x_1, x_2, x_3, \dots\}.$$

(iv) If S is a set of elements from some domain D satisfying property P, we can denote this set as:

$$S = \{x \in D | P(x)\}.$$

We illustrate with some examples.

Example 1.2. Determine which of the following sets are the same set:

(i) $\{x \in \mathbb{R} | x^2 \leqslant 1\}$

(ii) $\{-1, 0, 1, -1, 0\}$

(iii) $\{x \in \mathbb{R} | -1 \leqslant x \leqslant 1\}$

(iv) $\{x \in \mathbb{Z} | x^2 \leqslant 1\}$

(v) $\{\{x \in \mathbb{Z} | x^2 \leqslant 1\}\}$.

Notice that sets (i) and (iii) contain all real numbers between -1 and 1, hence they are the same set. Also, sets (ii) and (iv) both contain the integers $-1, 0$ and 1 and not other elements, so these sets are the same (note that the repetition of the elements of (ii) is irrelevant, as is the ordering). Finally, on first glance, it looks like (v) is also the same as (ii). However, it is not since (v) is the set which contains the set of integers $-1, 0$ and 1, and not simply the set of integers $\{-1, 0, 1\}$ - these are very different things. Other than those stated above, all other sets are mutually different.

As in logic, in many instances in set theory, the problems being considered are with respect to a particular fixed set (like the real numbers or complex numbers). To reflect this common situation, we need the following definition:

Definition 1.3. In the case where the underlying set to a mathematical problem is already presupposed, we call that set **a universal set** or **a universe of discourse** for the problem.

2. Subsets

Now we have the basic notation, we can start to consider sets related to given sets, and how sets relate to each other. Two of the most important elementary concepts in set theory are the notions of subset and set equality. We define them as follows:

Definition 2.1. If A and B are sets, we call A a **subset** of B written $A \subseteq B$ if an only if every element of A is an element of B i.e.

$$A \subset B \iff \forall x, x \in A \to x \in B.$$

We say A is a **proper subset** of B and write $A \subset B$ if and only if A is a subset of B, but there is an element of B which is not in A. We say A is **not a subset** of B and write $A \nsubseteq B$ if and only if there is an element in A which is not in B.

Definition 2.2. We say two sets A and B are equal and write $A = B$ if and only if every element of A is and element of B and every element of B is an element of A. Symbolically

$$A = B \iff A \subset B \wedge B \subseteq A.$$

One way to establish set containment is through the use of Venn Diagrams. Specifically, we draw a circle representing each set we are considering - if they share elements, we make the circles intersect, if they are distinct, we make the circles not intersect, and if one is a subset of the other, we draw the circle representing the subset inside the other. We shall make use of Venn diagrams later in this section and more generally, in this Chapter. We consider a couple of examples.

Example 2.3. Answer the following:

(i) Is $3 \in \{1, 2, 3\}$?

Yes, 3 is an element of this set.

(*ii*) Is $3 \subseteq \{1, 2, 3\}$?

No, 3 is not a set which contains any of the numbers 1, 2 or 3, so it cannot possibly be a subset.

(*iii*) Is $\{1\} \subseteq \{1, \{2\}\}$?

Yes, the set $\{1\}$ contains the number 1, and only the number 1, which is also in the set $\{1, \{2\}\}$.

(*iv*) Is $\{1\} \in \{1, \{2\}\}$?

No, the set $\{1\}$ is not in this set (though the number 1 is).

Remark 2.4. Note that there is a big difference between the symbols $\in$ and $\subseteq$ as illustrated in the last example.

Example 2.5. Determine which of the following sets are equal, subsets of another or not subsets of each other:

(*i*) $S = \{x \in \mathbb{R} | x^2 \leqslant 1\}$
(*ii*) $T = \{-1, 0, 1, -1, 0\}$
(*iii*) $U = \{x \in \mathbb{R} | -1 \leqslant x \leqslant 1\}$
(*iv*) $V = \{x \in \mathbb{Z} | x^2 \leqslant 1\}$
(*v*) $W = \{\{x \in \mathbb{Z} | x^2 \leqslant 1\}\}$.

We already considered this example. We know $S = U$ and $T = V$. We since every integer is a real number, it also follows that $T \subseteq S$, $T \subseteq U$, $V \subseteq S$ and $V \subseteq U$ (in fact they are proper subsets). Also, since S contains more elements that T, we have $T \nsubseteq U$ with similar relations holding for the other sets. Finally, W shares no elements with any other set, so $W \nsubseteq T$ etc.

Example 2.6. Let $A = \{n \in \mathbb{Z} | n = 6r - 5 \text{ for some integer r}\}$ and $B = \{m \in \mathbb{Z} | m = 3s + 1 \text{ for some integer s}\}$ Determine whether one is a subset of the other.

First, if $A \subseteq B$, then every element of A must lie in B. Suppose x is an arbitrary element of A. Then $x = 6r - 5$ for some integer r. Using simple arithmetic, we have

$$x = 6r - 5 = 6(r - 1) + 1 = 3(2r - 2) + 1 = 3k + 1$$

where $k = 2r - 2$. In particular, $x = 3k + 1$ for some integer k, and must lie in B, thus $A \subseteq B$.

Now we need to determine whether $B \subseteq A$. Before we do this, we consider a couple of numbers in B. Note that $4 \in B$. However, $4 \notin A$. To see this, suppose that $4 \in A$. Then $4 = 6r - 5$ for some integer r. But then $9 = 6r$ and thus $r = 9/6 = 3/2$ which is not an integer, contradicting that r is an integer. Thus $4 \notin A$ and hence $B \nsubseteq A$.

3. Operations on Sets

As with elementary arithmetic, there are some basic operations which can be performed on sets. The four basic operations are:

Definition 3.1. Let A and B be subsets of some universal set U. Then we define:

(i) the union of A and B denoted $A \cup B$ is the set of all elements $x \in U$ such that either $x \in A$ or $x \in B$:

$$A \cup B = \{x \in U | x \in A \vee x \in B\}$$

(ii) the intersection of A and B denoted $A \cap B$ is the set of all elements $x \in U$ such that $x \in A$ **and** $x \in B$:

$$A \cap B = \{x \in U | x \in A \wedge x \in B\}$$

(iii) the difference of A minus B, or the relative complement of B in A, denoted $A - B$ is the set of all elements $x \in U$ such that $x \in A$ but $x \notin B$:

$$A \cup B = \{x \in U | x \in A \wedge x \notin B\}$$

(iv) the complement of A in U denoted A^c is the set of elements $x \in U$ such that $x \notin A$:

$$A^c = \{x \in U | x \notin A\}$$

Note that if A and B have no elements in common, then their intersection does not contain any elements. This motivates the following two definitions:

Definition 3.2. We define the empty set, denoted $\not{o}$ to be the set containing no elements.

Definition 3.3. Two sets A and B are called disjoint if and only if they have no common elements:

$$A \text{ and } B \text{ are disjoint} \iff A \cap B = \emptyset$$

We consider some example to illustrate.

Example 3.4. Let $U = \mathbb{R}$ and let $A = \{x \in \mathbb{R} | 0 < x \leqslant 2\}$, $B = \{x \in \mathbb{R} | 1 \leqslant x \leqslant 4\}$, and $C = \{x \in \mathbb{R} | 3 \leqslant x \leqslant 9\}$. Answer the following:

(i) Find $A \cup B$

We have

$$A \cup B = \{x \in \mathbb{R} | 0 < x \leqslant 2 \vee 1 \leqslant x \leqslant 4\} = \{x \in \mathbb{R} | 0 < x \leqslant 4\}$$

(ii) Find $A \cap B$

We have

$$A \cap B = \{x \in \mathbb{R} | 0 < x \leqslant 2 \wedge 1 \leqslant x \leqslant 4\} = \{x \in \mathbb{R} | 1 \leqslant x \leqslant 2\}$$

(iii) Find $B^c \cap C$

We have

$$B^c = \{x \in \mathbb{R} | x \notin B\} = \{x \in \mathbb{R} | x < 1 \vee x > 4\}.$$

Therefore,

$$B^c \cap C = \{x \in \mathbb{R} | (x < 1 \vee x > 4) \wedge 3 \leqslant x \leqslant 9\} = \{x \in \mathbb{R} | 4 < x \leqslant 9\}$$

(iv) Find $A \cap C$

We have

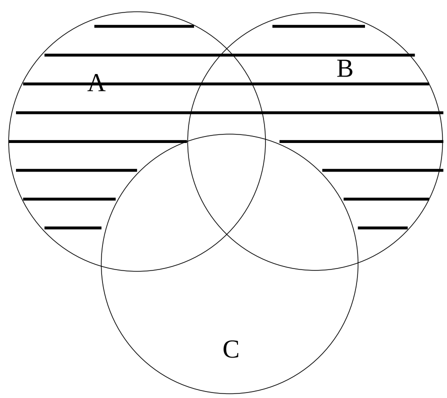
$$A \cap C = \{x \in \mathbb{R} | 0 < x \leqslant 2 \wedge 3 \leqslant x \leqslant 9\} = \emptyset$$

(v) Which pairs of sets are disjoint?

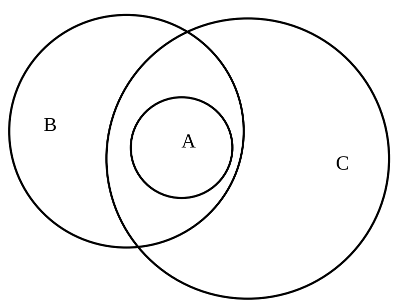
Two sets are disjoint if their intersection is the empty set. In this case, the only sets whose intersection is the empty set are A and C, and thus these are the only pair of sets which are disjoint.

As mentioned earlier, Venn Diagrams are useful when describing how sets relate to each other, especially when we are considering operations on sets. We illustrate a couple of examples.

Example 3.5. Shade in the part of the Venn diagram below which corresponds to the set $(A \cap B) \cup C^c$:



Example 3.6. Sketch a Venn diagram for three sets A, B and C satisfying the following: $A \subseteq B$, $A \subseteq C$, $C \nsubseteq B$ and $B \nsubseteq C$.



One useful operation in mathematics is to be able to break up a given set into a union of smaller sets, each of which share no elements with any other set e.g. breaking up the real numbers into positive reals, negative reals and 0. This process is so important that we shall give it its own definition. In order to do this, we need the following definition.

Definition 3.7. Sets $A_1, \dots A_n$ are said to be **mutually disjoint** if and only if no two sets share any elements i.e. $A_i \cap A_j = \emptyset$ unless $i = j$.

Definition 3.8. A collection of nonempty sets $\{A_1, \dots, A_n\}$ are called a partition of a set A if an only if both the following hold:

(i) $A = A_1 \cup \cdots \cup A_n$
(ii) $A_1, \dots, A_n$ are mutually disjoint.

We consider a couple of examples.

Example 3.9. Is the collection of sets $\{\{1,3\},\{2,5,7\},\{8,9\},\{4\},\{6\}\}$ a partition of the set $S = \{1,2,3,4,5,6,7,8,9\}$?

Yes - they are mutually disjoint and their union is S.

Example 3.10. Are the sets $\mathbb{Z}^+$ and $\mathbb{Z}^-$ a partition of the integers?

No - though they are mutually disjoint, neither contains 0, so their union is not $\mathbb{Z}$.

One last set operation to build a new set from old is the power set operation - specifically, if we are given a set, we can form a new set which consists of all the subsets of that set. The formal definition of this set is as follows:

Definition 3.11. Given a set A, the power set of A denoted $\mathcal{P}(A)$ is the set of all subsets of A.

We consider a couple of examples:

Example 3.12. Determine the power set of $S = \{x, y, z\}$.

We have

$$\mathcal{P}(S) = \{\emptyset, \{x\}, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$$

Note that the power set contains 8 elements.

Example 3.13. If a set has n elements, how big is the power set? Why?

It will have size 2^n - this can be proved directly or using induction. To prove it directly, observe that any subset is determined by its elements, and given any x in a set, it either lies in a subset or doesn't. This means there are 2^n total choices to be made when deciding which elements lie in a given subset, and each possible such choice will lead to a distinct subset.

The last operation we can perform on sets is constructing their Cartesian products which is a generalization of 2-space, $\mathbb{R}^2$.

Definition 3.14. Given two sets A and B, we define the Cartesian product of A and B denoted $A \times B$ to be the set of all ordered pairs

(a, b) where $a \in A$ and $b \in B$ with the notion that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
More generally, given sets $A_1, \ldots, A_n$, we define the Cartesian product of $A_1, \ldots, A_n$ denoted $A_1 \times \cdots \times A_n$ to be the set of all ordered n-tuples $(x_1, \ldots, x_n)$ where $x_i \in A_i$ for $1 \leqslant i \leqslant n$.

Though this concept is probably very familiar, we finish with a simple example.

Example 3.15. Let $A = \mathbb{R}$ and $B = \{0, 1\}$. Write down a general element of $A \times B$.

Elements of $A \times B$ will either be of the form $(x, 0)$ or $(x, 1)$ for some real number x.

Homework

(*i*) From the book, pages 267-269: Questions: 3, 5, 7, 8c, 8d, 8g, 9c, 9d, 9e, 11a, 11c, 11g, 11h, 11i, 11j, 12a, 12c, 12h, 13a, 14, 17, 18, 21d, 21e, 21b, 21c, 25, 28, 29