Section 5.2: Properties of Sets

In this section we shall consider some of the basic properties of sets. First, we shall consider some elementary properties of how certain sets related to each other. The main purpose of this is to allow us to develop techniques to prove more difficult results. Following this, we shall state some more general set relations (such generalizations of De Morgans laws, commutative and distrubutive laws for sets) and then prove some of them. We shall finish the section by examining further properties of the empty set.

1. Basic Subset Relations and Element Arguments

In general, if we want to prove that X is a subset of Y, we use the following method:

Result 1.1. (Element Argument) To show $X \subseteq Y$, we do the following:
suppose $x \in X$ is a particular but arbitrarily chosen value of x
show x is an element of y

We illustrate.

Theorem 1.2. *The following relations hold for all sets A, B and C:*

- (i) *(Inclusion of intersection)*
  - (a) $A \cap B \subseteq A$
  - (b) $A \cap B \subseteq B$
- (ii) *(Inclusion in Union)*
  - (a) $A \subseteq A \cup B$
  - (b) $B \subseteq A \cup B$
- (iii) *(Transitivity) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. We shall illustrate how to prove such statements by proving (iii). Suppose that $x \in A$. We need to show $x \in C$. Since $x \in A$ and $A \subseteq B$, it follows that $x \in B$. Since $x \in B$ and $B \subseteq C$, it follows that $x \in C$. Thus if $x \in A$, then $x \in C$ and so $A \subseteq C$.

□

To show that two sets are equal, we would take the following steps:

Result 1.3. Let X and Y be two sets. T prove $X = Y$, we do the following:

- (i) show that $X \subseteq Y$
- (ii) show that $Y \subseteq X$

When proving sets are equal, the following formal interpretations of what it means to lie in certain sets will be very useful:

Definition 1.4. Let X and Y be subsests of U and suppose $x, y \in U$.

(i) $x \in X \cup Y \iff x \in X$ or $x \in Y$
(ii) $x \in X \cap Y \iff x \in X$ and $x \in Y$
(iii) $x \in X - Y \iff x \in X$ and $x \notin Y$
(iv) $x \in X^c \iff x \notin X$
(v) $(x, y) \in X \times Y \iff x \in X$ and $y \in Y$

We illustrate with an example.

Example 1.5. Show that $A \cup (A \cap B) = A$.

First we show that $A \subseteq A \cup (A \cap B)$. Suppose that $x \in A$. Then $x \in A \cup (A \cap B)$ since $x \in A \cup (A \cap B)$ if and only if either $x \in A$ or $x \in A \cap B$ (and we are assuming $x \in A$). Thus $A \subseteq A \cup (A \cap B)$.

Now suppose $x \in A \cup (A \cap B)$. Then either $x \in A$ or $x \in A \cap B$. If $x \in A$, then we are done. If $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular, $x \in A$. Therefore, in both cases, $x \in A$, and thus $x \in A \cup (A \cap B)$ implies $x \in A$ so $A \cup (A \cap B) \subseteq A$.

2. Set Identities

There are a number of very important set identities which we can derive. The identities are listed in a table on page 272 (we shall not list them here). We shall derive some of these identities for ourselves and then illustrate how these identities can be used to derive further identities using "algebraic" style proofs.

Example 2.1. Prove the distributive set identity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose that $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, and thus $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$, so $x \in A \cap B$, $x \in A \cap C$ and consequently $x \in (A \cup B) \cap (A \cup C)$.

Now suppose $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$ and we are done. If $x \notin A$, then since $x \in A \cup B$, we must have $x \in B$ and likewise, since $x \in A \cup C$ we have $x \in C$. Thus $x \in B \cap C$ and hence $x \in A \cup (B \cap C)$.

Example 2.2. Prove the De Morgan law $(A \cup B)^c = A^c \cap B^c$.

Suppose that $x \in (A \cup B)^c$. Then $x \notin A \cup B$ and consequently $x \notin A$ and $x \notin B$. It follows that $x \in A^c$ and $x \in B^c$ and so $x \in A^c \cap B^c$.

Suppose that $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. It follows that $x \notin A$ and $x \notin B$ and so $x \notin A \cup B$. Thus $x \in (A \cup B)^c$.

Example 2.3. Use set identities to show that

$$(A - C) \cap (B - C) \cap (A - B) = \emptyset$$

First, we use the set different laws three times:

$$(A-C)\cap(B-C)\cap(A-B)=(A\cap C^c)\cap(B\cap C^c)\cap(A\cap B^c)$$

Now using the commutative and associative laws. we have

$$(A\cap C^c)\cap(B\cap C^c)\cap(A\cap B^c)=(A\cap A)\cap(C^c\cap C^c)\cap(B\cap B^c)$$

Next, using Idempotent laws and complement laws, we get

$$(A\cap A)\cap((C^c\cap C^c)\cap(B\cap B^c))=A\cap(C^c\cap\emptyset)$$

Finally, using the universal bound law, we get

$$A\cap(C^c\cap\emptyset)=A\cap\emptyset=\emptyset.$$

3. The Empty Set

We finish by looking at some important properties of the empty set. The first property we prove is that an empty set is a subset of every possible set. We shall then show that there is only one empty set) and hence referring to it as "the" empty set as we have been doing makes sense).

Theorem 3.1. *If E is a set with no elements and A is any other set, then $E\subseteq A$.*

Proof. We shall prove this result by contradiction. Suppose that this is not the case. Then it follows that there exists an element x such that $x\notin A$ but $x\in E$. However, there are no elements in E, so this is a contradiction. Thus we must have $E\subseteq A$. □

Corollary 3.2. *There is only one set with no elements*

Proof. Suppose E_1 and E_2 are sets with no elements. We shall show $E_1=E_2$ using Theorem 3.1. Note that since E_1 contains no elements and E_2 is a set, we must have $E_1\subseteq E_2$. Likewise, since E_2 contains no elements and E_1 is a set, we must have $E_2\subseteq E_1$. Thus it follows that $E_1=E_2$. □

In general, when we are trying to show a given set is empty, we take the following steps:

Result 3.3. In order to show a set A is empty, we do the following:

(i) Suppose x is an element of A
(ii) Derive a contradiction under this assumption

We finish with an example.

Example 3.4. Show that if O is the set of odd integers and E is the sets of even integers, then $E\cap O=\emptyset$.

Suppose not. Then there exists an integers n which is both even and odd. Since n is even, $n=2k$, and since n is odd, $n=2m+1$. Thus we have $2k = 2m + 1$ and so $2(k - m) = 1$ giving $k - m = \frac{1}{2}$. Since k and m are integers, $k - m$ is an integer. However, $\frac{1}{2}$ is not an integer, and since $k - m = \frac{1}{2}$, it follows that $k - m$ is not an integer. This is clearly a contradiction, and so such an n cannot exist. Thus $E \cap O = \emptyset$.

Homework

(i) From the book, pages 280-282: Questions: 1, 4, 6, 8. 11, 14, 16, 18, 21a, 21d, 25, 30, 34