Section 5.4: Russell's Paradox

At the turn of the twentieth century, there was a movement in philosophy and mathematics to attempt to formalize all of mathematics - specifically, people wanted to set up a solid foundation on which all mathematics can be built. The approach taken was to build mathematics up from set theory. Using not much more than we have already considered, there is a way to define the integers, and from this, the operations of multiplication and addition. With these operations defined, we can then construct the real number system and consequently, differential calculus. This ambitious project was pursued by a number of mathematicians and philosophers, most notably Frege, Russell and Whitehead. This project came into a major stumbling block however when Russell showed that axiomatic set theory gave rise to certain paradoxes (a paradox is a sentence which does not have a well defined truth value - if it is true, then it is false, and if it is false, then it is true). In this section, we shall consider Russell's paradox and other examples of paradoxes in logic.

1. Russell's Paradox

An assumed well defined principle in naïve set theory is that of set membership - specifically, given an object and a set, that object either lies in that set or does not lie in that set (i.e. $P \vee \sim P$). Russell's paradox shows that this is not the case at all, and in fact there are sets which can be constructed and objects for which they must lie and not lie in that given set (hence a paradox, or contradiction). The construction is as follows.

Example 1.1. Let U be the universe which consists of all possible sets. Consider the following set:

$$S = \{A | A \notin A\}$$

In words, S is the set of sets which are not members of themselves. For example, the set of all integers is not itself an integer, and hence the set of integers lies in this set (since $\mathbb{Z} \notin \mathbb{Z}$). Alternatively, the set of all things which are not Chihuahuas is a member of itself since it is not a Chihuahua and hence does not lie in S. Thus there are examples of sets which lie in S and examples of sets which do not like in S. Russell's paradox lies in the simple question "is S a member of S?"

We consider the two possibilities. First, if $S \in S$, then S is a member of itself i.e. $S \in S$. However, S consists of all sets which are not members of themselves and thus if $S \in S$, then $S \notin S$ which is clearly absurd. Alternatively, if $S \notin S$, then S satisfies the defining property for set membership in S, so $S \in S$, again, an absurd consequence. Hence S is neither a member or not a member of S, which of course is a paradox since it must either be a member of S or not a member of S.

2. Distinguishing between Paradoxes and Statements

Recall that a statement is a sentence which is either true or false. There are many examples of sentences which can at first seem paradoxical (and hence not statements) but are indeed statements (since they are either true or false), and likewise, there can be other statements which are paradoxes like Russell's paradox. To illustrate, we consider some examples.

Example 2.1. Which of the following are statements:

(i) All real numbers with negative squares are primes

In order to determine whether this is a statement, we need to decide whether or not is has a well defined truth value. Note that this is a universal conditional i.e. more formally, this reads "for all real numbers r, if r^2 is negative, then r^2 is prime". Observe that since the hypothesis of this conditional is false (since the square of any real number is never negative), the conditional is vacuously true. This the sentence is true and hence it is a statement (all be it, meaningless!)

(ii) This sentence is false or $1 + 1 = 3$

This is a conjunction statement - it is false if both statements are false and true if either is true. We know $1+1 = 3$ is always false, so the truth value of this sentence will depend upon the sentence "this sentence is false". If this sentence is true, then it is false. Likewise, if this sentence is false, then it must be true. In particular, the first sentence does not have a well defined truth value, so is not a statement, and hence the conjunction is not a statement either.

Homework

(i) From the book, pages 296: Questions: 1, 2, 3, 5

(ii) Determine another logical paradox and explain why it is a paradox (you can use references to find a paradox, but should reference when you do)