Section 6.1: Introduction to Probability

On very important subfield of mathematics is probability. It is used in many different real life situations such as determining insurance premiums, or counting the different number of pizzas which can be constructed given a certain number of possible toppings. In this chapter, we shall introduce some of the important ideas in probability and consider a number of elementary applications. In addition, we shall consider the related problem of counting elements from a set (which is very important when we are considering the probability of something occuring). The main goal of this chapter is preparation for a more advanced course in probability and statistics (which many of you will be taking in future years).

1. Basic Definitions in Probability

We start with some basic definitions and facts. For those already enrolled in a statistics and probability class, much of this will probably be very familiar to you (though we may define some things slightly differently).

Definition 1.1. A random process is a process with the property that we know one outcome from a fixed set of outcomes must occur, but it is impossible to predict withe certainty what outcome will occur.

Definition 1.2. The sample space of a random process is the set of all possible outcomes of that process.

Definition 1.3. An event of a random process is a subset of the sample space.

We illustrate with an example.

Example 1.4. Suppose you roll a fair 6-sided dice and flip a fair coin at the same time. What is the sample space of possible outcomes? Also, write down all the elements in the event "a 1 is rolled" and "either a tail is flipped or a 2 is rolled".

The sample space will be the following set of twelve pairs of possible outcomes (the number represents the roll on the die, and T and H represent the heads or tail outcome):

$$\left\{ \begin{array}{c} (1,H),(1,T),(2,H),(2,T),(3,H),(3,T), \\ (4,H),(4,T),(5,H),(5,T),(6,H),(7,T) \end{array} \right\}$$

The event "a 1 is rolled" is the subset

$$\{(1,H),(1,T)\}$$

The event "either a tail is flipped or a 2 is rolled" is the following subset:

$$\{(1,T),(2,H),(2,T),(3,T),(4,T),(5,T),(7,T)\}$$

The following notation will be useful.

Notation 1.5. If A is a set with a finite number of elements, then we define $N(A)$ to be the number of elements of A.

Now we have the necessary definitions to define the probability of a given event occurring.

Definition 1.6. Suppose that S is a sample space for some random process in which all outcomes are equally likely and suppose that E is an event in S. Then the probability of E denoted $P(E)$ is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S} = \frac{N(E)}{N(A)}.$$

2. Simple Examples in Probabilities

We illustrate our basic definitions with some simple (and famous) examples in probability.

Example 2.1. Suppose the sample space of a random process is a deck of cards. Determine the probabilities of the following events:

(i) Drawing a king from the deck

There are 52 total cards and 4 kings. Thus the probability of drawing a king is $4/52 = 1/13$

(ii) Drawing a heart from the deck

There are 52 total cards and 12 hearts. Thus the probability of drawing a heart is $13/52 = 1/4$

Example 2.2. (The Monty Hall Problem) You are on a game show and there are three doors, say A, B and C. Behind one of the doors is the grand prize, and behind the other doors is nothing. You need to pick which door you are going to open. Lets say you pick door A. After doing this, the game show host opens one of the doors, say door B to reveal nothing. The game show host then asks if you would like to stick with the door you originally choose, or switch to the other one. What should you do?

This is a classic problem in probability. It seems that your probability would be the same regardless, so it wouldn't matter if you switch. However, with a little deeper analysis, we see that this is not necessarily the case. Lets look at the different possible scenarios.

First there are three different possibilities: Case 1: the prize is behind door A; Case 2: the prize is behind door B; Case 3: the prize is behind door C. Let the sample space consist of these three equally likely possibilities and consider the outcome E="you switch and win". We have the following possibilities:

(i) Case 1 - if the prize is behind door A and you switch doors, you will lose.
(ii) Case 2 - if the prize is behind door B and you switch doors, you will win (since door C would have been opened).
(iii) Case 3 - if the prize is behind door C and you switch doors, you will win (since door B would have been opened).

This means for two of the three possible cases, you will win if you switch doors. Thus the probability of winning if you switch doors is 2/3.

3. Basic Problems in Counting

As we saw in the previous problems, much of the work when determining probabilities is that of counting the number of elements in a given list. In the following sections, as we consider problems in probability, we shall also develop techniques to help us count the number of elements in a given list. We finish with a couple of explicit counting problems. The first, though obvious, will be very useful for us.

Theorem 3.1. *If m and n are integers and $m \leqslant n$, then there are $n - m + 1$ integers from m to n inclusive.*

Proof. The proof is straight forward if we list the integers between m and n as m plus other integers. Specifically, we have

$$m, m+1, m+2, \ldots m+(n-m-1) = n-1, m+(n-m) = n.$$

Notice that the number of integers will be equal to listed from m to $m + r$ up to and including $m + r$ will be equal to $r + 1$. In particular, the number of integers between m and n and including m and n will be $n - m + 1$.

□

We finish with an example.

Example 3.2. What is the probability that a randomly chosen positive three digit integer will be divisible by 6?

In this case, out sample space is the set of all positive three digit numbers and our event is that a number is picked which is divisible by 6. Therefore, to work out the probability, we need to calculate the following two things:

(i) the size of the sample space
(ii) the size of the set of all three digit numbers divisible by 6

Using our formula, the number of three digit numbers will be equal to $999 - 100 + 1 = 900$. Next we need to work out the number of three digit numbers divisible by 6. To do this, we line up all three digit numbers and indicate which are multiples of 6:

100	101	102	103	104	105	106	107	108	$\cdots$	995	996	997	998	999
		$17 \cdot 6$						$18 \cdot 6$	$\cdots$		$166 \cdot 6$			

By writing them out in this way, it is clear that the number of positive three digit integers divisible by 6 will be equal to the number of integers between 17 and 166 (inclusive), and thus there are $166 - 17 + 1 = 150$. Therefore, the probability of picking a 3 digit number which is divisible by 6 is

$$\frac{150}{900} = \frac{1}{6}.$$

Homework

(*i*) From the book, pages 304-306: Questions: 2, 3, 8, 11, 13, 14, 15, 18, 21,