Section 6.2: Possibility Trees and the Multiplication Rule

Our next goal is to consider probabilities of different outcomes from a sequence of events occurring in a specific order. In order to determine such probabilities, we shall introduce the idea of a probability tree which will allow us to track all different possible outcomes. These ideas will provide new ways to count the number of elements in a sample space and hence will allow us to determine probabilities of different outcomes of the sequence of events. We shall finish by using our observations to help determine formulas to count the number of permutations of elements from a given set.

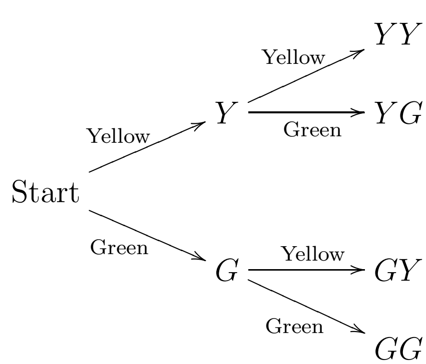
1. Probability Trees and the Multiplication Rule

We define a probability tree to track outcomes of a sequence of events as follows:

Definition 1.1. Suppose that there is a sequence of events occurring in a specific order. Then, starting at a point, we draw a line out from that point for all possible outcomes of the first event. From the end of each of these lines, we then draw a line for each possible outcome from the next event and so on until we reach the final outcome of all events. We call such a diagram a possibility tree for that sequence of events.

The definition of a probability tree seems complicated, but it is actually fairly easy. We illustrate with an example.

Example 1.2. Suppose that there are two cups, each containing an equal number of yellow and green balls. You take one ball from one cup and then one ball from the other cup. Sketch the probability tree to determine all possible outcomes and then determine the probability that two balls of the same colour are drawn.



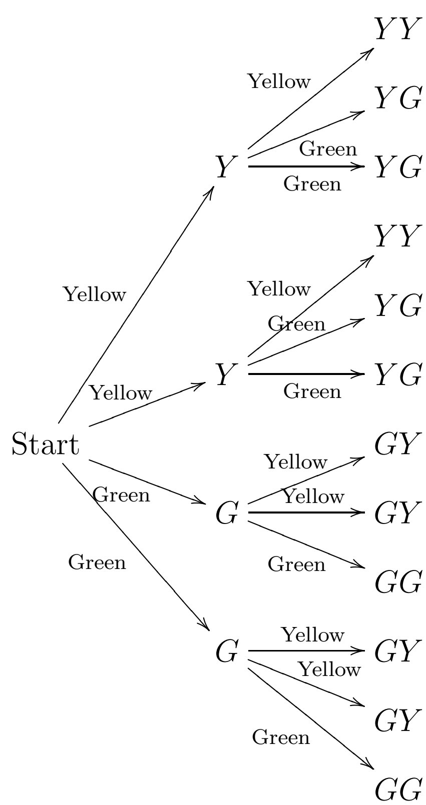
Technically speaking, we need a branch of the tree for every possibility (meaning if there are 50 green and 50 yellow balls in the first cup, there are 100 possibilities, and similarly with the second set of branches).

However, since three are equal numbers of balls in each cup, we note that it is equally likely that either are drawn in both drawings, and so we restrict branches to the two possibilities. This gives tree we sketched above.

From the probability tree, we see that there are a total of 4 possible outcomes (each of which are equally likely). Two of these outcomes result in two balls of the same colour being drawn, and thus there is a 1/2 probability of two balls of the same colour being drawn.

Example 1.3. Now suppose there is one cup containing two yellow and two green balls. You take one ball from the cup and then take another from the same cup. Sketch the probability tree to determine all possible outcomes and then determine the probability that two balls of the same colour are drawn.

In this case we need to be a little more careful since the second drawing will depend upon th first. Specifically, if we draw a green ball, then on the second draw, we will have two possibilities where a yellow ball is drawn as opposed to one possibility that a green ball is drawn. To illustrate the possibilites, we need a bigger tree diagram, with a branch to represent each possible choice.



From the probability tree, we see that there are a total of 12 possible outcomes (each of which are equally likely). Four of these outcomes result in two balls of the same colour being drawn, and thus there is a 1/3 probability of two balls of the same colour being drawn.

These two examples suggest the following method to count the number of possible outcomes which is the consequence of a sequence of events.

Theorem 1.4. *If a process consists of k steps, and*

- *the first step can be performed in n_1 ways*
- *the second step can be performed in n_2 ways (regardless of how the first step was performed)*

$$\vdots$$

- *the kth step can be performed in n_k ways (regardless of how all previous steps were performed)*

Then the whole process can be completed in $n_1 \cdot n_2 \cdots n_k$ different ways.

We illustrate with an example.

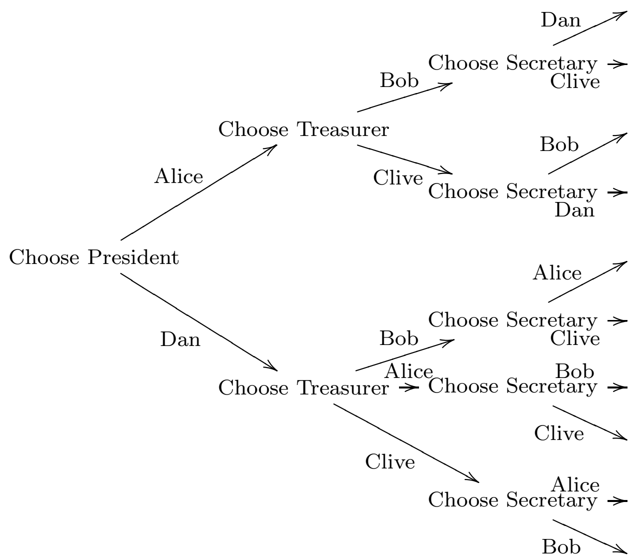
Example 1.5. How many numbers between 1 and $99,999$ contain exactly one of each of the digits $2, 3, 4$ and 5?

We shall interpret this as a process of constructing a five digit number (where we make any number which has less than five digits a number with five digits by adding zeros to the front i.e. $1 = 00001$). Since any such number must contain each of $2, 3, 4$ and 5, each event will be placing these numbers as one of the digits in the number. First, for 2, there are 5 places, for 3, there will be four, for 4 there will be three and for 5 there will be two remaining places. For the last digit, we can choose any of $0, 1, 6, 7, 8, 9$, so there will be six total choices. Thus, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 6 = 720$ total such numbers between 1 and $99,999$.

Remark 1.6. We note that the condition "regardless of how all previous steps were performed" is necessary. Specifically, if making any choice during a process adjusts the number of choices in the next step of the process, then the formula does not hold. Under such circumstances, we would need to sketch a probability tree to calculate probabilites. We illustrate with an example.

Example 1.7. Suppose that three officers, a president, treasurer and secretary, must be chosen from four people: Alice, Bob, Clive and Dan. However, Dan does not have the qualifications to be treasurer and Neither Clive nor Bob have the time to act as president. Determine the number of possible ways the officers can be chosen and decide who is most likely to be president.

On first glance, it may look like we can use the multiplication rule. Specifically, there are two choices for president, three remaining possibilities for secretary, and then two for treasurer. However, notice that the number of choices for treasurer will result affect the different choices for secretary, so we cannot use the multiplication rule. Instead, we need to use a tree. We have the following:



Thus there are 10 different possibilities for the choice of the officers. Notice that 6 of these possibilities have Dan as president and only 4 have Alice as president, and thus it is more likely that Dan will be president (a 3/5 probability).

2. Permutations

We shall now consider applications of our observations from the last section. First, we need the following definition:

Definition 2.1. A permutation of a set of objects is an ordering of those objects in a row.

We illustrate with an example.

Example 2.2. Write down all permutations of the set $\{a, b, c\}$.

We have abc, acb, bac, bca, cab, and cba, so there are a total of 6 permutations of the set $\{a, b, c\}$.

We can use our observations from the previous section to prove the following Theorem.

Theorem 2.3. *For any integer n with $n \geqslant 1$, the number of permutations of a set with n elements is $n!$.*

Proof. We can think of a permutation as a process where each step, we choose an element remaining from the set which has not already been chosen, and write it in the row. For the first step, there are n choices, for the second step, $n-1$ choices and so on. Using the multiplication rule, we get that the total number of outcomes is $n \cdot (n-1) \cdots 2 \cdot 1 = n!$.

□

We illustrate with an example.

Example 2.4. How many different ways are there of reordering the letters in the sentence "Math is fun"?

Since no letters are repeated, the number of different ways to reorder the letters in this sentence will simply be equal to the number of different permutations of the letters of this sentence i.e. $9! = 362,880$.

A closely related problem to permutations is that of choosing in a particular order a certain number of elements from a given set. Formally, we define it as follows:

Definition 2.5. An r-permutation of a set of n elements is an ordered selection of r elements taken from that set of n elements. The number of r-permutations of a set of n elements is denoted $P(n, r)$.

We have the following way to count r permutations.

Theorem 2.6. *If n and r are integers and $1 \leqslant r \leqslant n$, then the number of r permutations of a set of n elements is given by the formula*

$$P(n, r) = n \cdot (n-1) \cdot \cdots \cdot (n-r+1)$$

or equivalently

$$P(n, r) = \frac{n!}{(n-r)!}$$

Proof. As with previous proofs, we consider choosing elements to put in a row as a process. There are n choices for the first elements, $n-1$ for the second and so on. Since there are a total of r elements we are choosing, for the last choice, there will be $n-r+1$ elements to choose from. Thus using the multiplication rule, we have

$$P(n, r) = n \cdot (n-1) \cdot \cdots \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

□

We finish with a couple of examples of how to use this formula.

Example 2.7. How many different ways can five letters of the sentence "Math is fun" be selected when written in a row if the third letter is M.

Since the third letter is already fixed, we are simply asking of the remaining elements, how many different ways are there of selecting four of them in a row. Since there are only 8 letters left, this will be equal to

$$P(8, 4) = \frac{8!}{(8-4)!} = \frac{8!}{(4)!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

Example 2.8. Prove that for all integers $n \geqslant 3$,

$$P(n+1,3) - P(n,3) = 3P(n,2)$$

In order to prove this equality, we simply need to evaluate both sides of the equation and check that there are equal. For the left hand side, we have

$$P(n+1,3) - P(n,3) = \frac{(n+1)!}{(n-2)!} - \frac{n!}{(n-3)!}$$

$$= (n+1)(n)(n-1) - n(n-1)(n-2) = n(n-1)(n+1-(n-2))$$

$$= 3n(n-1).$$

For the right hand side, we have

$$3P(n,2) = \frac{3(n!)}{(n-2)!} = 3n(n-1)$$

Hence both sides are equal.

Homework

(i) From the book, pages 318-320: Questions: 5, 7, 9, 13, 14, 16, 17, 19, 29, 30, 31, 33, 35, 38