Section 6.4: Counting Subsets of a Set: Combinations

In section 6.2, we learnt how to count the number of r-permutations from an n-element set (recall that an r-permutation is an ordered selection of r elements from a set containing n elements). Note that a permutation depends upon ordering - that is, if $x, y \in S$, a set, then the permutation x, y is different to that of y, x. In this section, we introduce a similar notion to a permutation called a combination, where order does not matter i.e. so x, y and y, x would be considered the same. The idea of a combination is helpful when calculating probabilities of events where the order of choices does not matters (so hands in poker for example).

1. Combinations

We start with a definition.

Definition 1.1. Let n and r be non-negative integers with $r \leqslant n$. An r-combination of a set of n elements is a subset r of the n elements. The symbol $\binom{n}{r}$ which is read "n choose r" denotes the number of subsets of size r that can be chosen from a set of n elements.

Our first task is to determine a formula to calculate the number of r combinations of an n element set. We first consider an example to gain some insight into the problem.

Example 1.2. Determine the total number of 5 card hands in poker.

There are a total of 52 cards in a deck. The total number of 5 card permutations will be

$$P(52, 5) = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311,875,200.$$

It seems like this should be the total number of hands, but notice that a permutation gives an ordered set, and in a five card hand, we do not care about the order the cards are given in. To see this, consider one of these five card hands, say a royal flush in hearts, $10, J, Q, K, A$. Notice that there $P(5, 5) = 5!$ different permutations of this five card hand, none of which do not change the hand itself (since it is a simple reordering) e.g $10, J, Q, K, A$ and $J, 10, Q, K, A$ are both the same hand, they are just reordered. This is in fact true for any five card hand we choose. In particular, the total number of 5 card hands in poker will be the total number of permutations of 5 cards from 52 (so the number of ordered hands) divided by the number of different permutations of that hand - so

$$\binom{52}{5} = \frac{P(52, 5)}{P(5, 5)} = \frac{311,875,200}{120} = 2,598,960.$$

From the previous example, to calculate the number of subsets of size r from a set of size n, which are unordered permutations, we calculate the number of ordered sets of size r, $P(n, r)$, and then divide it by the total number of ways we could permute any one of these ordered sets, so $r!$. Thus,

Theorem 1.3. *The number of subsets of size $r \geqslant 0$ (or r-combinations) that can be chosen from a set of $n \geqslant r$ elements, $\binom{n}{r}$ is given by the formula*

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

We illustrate with some examples.

Example 1.4. How many different ways can a committee of 6 be selected from 15 people? What if there are two people who will only serve on the committee if both of them are on it?

For the first question, it is a straight forward application of the theorem. There are

$$\binom{15}{6} = \frac{15!}{6! \cdot 9!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 5005$$

For the second problem, it is a little more difficult because of the additional constraints on the choice of committee. To approach this problem, we shall break it up into two calculations - whether these two people are on the committee, or are not on the committee. We shall calculate the total number of committees on which these two people will be on, and then the total number of committees they will not be on, and the total number of committees will be the sum of these. First, if the two people are on the committee, the remaining 4 committee members will be chosen from the remianing 13 people. Thus there are

$$\binom{13}{4} = \frac{13!}{9! \cdot 4!} = 715$$

committees on which these two people could serve. If these people are not on a committee, than all members are chosen from the remaining 13 people, giving

$$\binom{13}{6} = \frac{13!}{7! \cdot 6!} = 1716$$

Therefore, there are a total of

$$\binom{13}{6} + \binom{13}{4} = 2431$$

committees satisfying these criteria.

Example 1.5. Determine the probability of a five card poker hand being a flush (all possible flushes including straight flushes).

The total number of five card hands will be the number of subsets of size five from the 52 card deck, so will equal

$$\binom{52}{5} = 2,598,960$$

The number of flushes will be equal to four times the number of flushes in any given suit. Since a flush is five cards of the same suit and there are thirteen cards of any one suit, it follows that there are

$$\binom{13}{5} = 1287$$

flushes in one suit, and thus 5148 total flushes. Thus the probability of a five card hand being a flush is

$$P(F) = \frac{N(F)}{N(U)} = \frac{5148}{2,598,960} = 0.001981.$$

Example 1.6. Determine the probability of a five card poker hand being a straight (all possible straights including straight flushes).

This problem is a little bit more difficult since it is not immediately obvious how to count the number of straights (since they can be of any suit). Since a straight is five consecutive cards, we can think of it as follows: we can count the number of straights with a given starting value (so the straights which start at 2, the straights which start at 3 and so on), and then multiple by the total number of starting values (of which there are 10, either, $A, 2, 3, 4, 5, 6, 7, 8, 9$ or 10). If a straight starts off at 2, that 2 could come from any of the four suits, so there are $\binom{4}{1} = 4$ different choices. Likewise, for the 3, 4, 5 and 6. Since each choice is independent of the previous choice, the multiplication rule says that the total number of straights starting at 2 will be $4^5 = 1024$. Since there are 10 different places to start a straight, and an equal number of straights regardless of where it starts, we have a total of 10240 straights. Thus the probability of a straight in a five card hand is

$$\frac{10240}{2,598,960} = 0.0039$$

2. Counting Permutations of Sets with Repeated Terms

The result we have developed for combinations can also be used to help solving problems regarding permutations of a given list where terms may be repeated. We illustrate with an example.

Example 2.1. How many distinguishable orderings of the word "WOOTTON" are there?

Observe that any permutation of the same letters will not give a different word, so we cannot simply calculate the permutations - we need to be careful about repeated letters. Therefore, we shall consider how to place each letter in order. First, we know there are a total of 7 letters, so 7 positions to write the letters. If we take the letter "W", it can be placed in any one of the 7 positions. Next, consider the "O"'s. There are 6 places left to write letters. The number of positions we can write the "O"'s will be equal to the number of subsets of the 6 remaining positions of size 3, so will be equal to $\binom{6}{3} = 20$. The number of positions we can write the "T"'s will be equal to the number of subsets of the 3 remaining positions of size 2, so will be equal to $\binom{3}{2} = 3$. Finally, the "N" will be placed in the last position. Since each of these steps did not depend upon the previous, we can use the multiplication rule, and thus we get the total number distinguishable orderings of the word "WOOTTON" is equal to

$$\binom{7}{1} \cdot \binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = 7 \cdot 20 \cdot 3 \cdot 1 = 420.$$

This example is actually a special case of a more general idea of when we are trying to determine permutations of n elements where some of them are repeated.

Theorem 2.2. *Suppose that a collection consists of n objects of which:*

- n_1 *of them are the same, but different from the others*
- n_2 *of them are the same, but different from the others*

$$\vdots$$

- n_k *are the same, but different from the others*

and suppose that $n_1 + \cdots + n_k = n$. Then the total number of distinct permutations of the n objects is

$$\binom{n}{n_1} \cdot \binom{n - n_1}{n_2} \cdots \binom{n - n_1 - n_2 - \cdots - n_{k-1}}{n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

3. Warnings and Further Examples

The tools we have developed to count sets are very useful, but they should always be used with caution. We have seen examples where we have calculated certain combinations and then added the results, and in other cases we have used the multiplication rule. This of course entirely depends upon the situation you are considering, and there is no one way to tell which rule should be used when. Another problem people sometimes encounter is counting certain objects too many times accidently. The best way to avoid these issues is to be very careful and always keep asking yourself the following two questions:

- Am I counting everything I should be?
- Is everything being counted only once?

We finish with a couple more examples.

Example 3.1. A five a side coed soccer team has 8 men and 5 women. At any given time, there must be at least two men and two women on the pitch. How may possible teams can be made up?

There must either be 2 men on the pitch, or 3 men, so we shall calculate the number of teams with 2 men, and then add the number of teams with 3 men. First, for a team containing 2 men, there are $\binom{8}{2} = 28$ different possibilities of choices for men and $\binom{5}{3} = 10$ different possibilities of choices of women. Since these choices do not affect each other, the multiplication rule says that there will be a total of

$$\binom{8}{2} \cdot \binom{5}{3} = 28 \cdot 10 = 280$$

teams with 3 women and 2 men. By a similar argument, there will be a total of

$$\binom{8}{3} \cdot \binom{5}{2} = 56 \cdot 10 = 560$$

teams with 2 women and 3 men. Thus there are a total of $560 + 280 = 860$ teams.

Example 3.2. At Macaroni Grill, you can create your own pasta by choosing one of six different pastas, one of seven sauces and up to four of eleven seasonal ingredients. How many total pasta dishes are there?

Clearly each choice (of sauce, of pasta and ingredients) is independent, so we can use the multiplication rule. There are $\binom{6}{1} = 6$ choices of pasta, and $\binom{7}{1} = 7$ choices of sauce. If someone chooses 0 ingredients to add, then there will be $7 \cdot 6 = 42$ total choices. If someone chooses one ingredient, they will have a total of $7 \cdot 6 \cdot \binom{11}{1} = 462$ choices. Likewise, if someone chooses two, three of four ingredients, they will have a total of $7 \cdot 6 \cdot \binom{11}{2} = 2310$, $7 \cdot 6 \cdot \binom{11}{3} = 6930$, or $7 \cdot 6 \cdot \binom{11}{4} = 13860$ choices. Therefore, there will be a total of

$$42 + 462 + 2310 + 6930 + 13860 = 23,604$$

different pasta dishes.

Homework

(*i*) From the book, pages 347-349: Questions: 2, 4, 5, 7, 9, 15, 16, 19, 24, 26, 28