

Section 7.1: Functions Defined on General Sets

In this chapter, we return to one of the most primitive and important concepts in mathematics - the idea of a function. Functions are the primary object of study in differential and integral calculus, and have many obvious applications both in mathematics and in the real world. In this section, we shall present a formal definition for a function and all related terminology and considered some specific examples.

1. THE DEFINITION OF A FUNCTION

Naively speaking, a function is a relationship between two quantities, one of which is completely determined by the value of the other. For example, your grade in this class is completely determined by your overall points achieved throughout the class, so your grade is a function of the points you have. Our first task is to move away from a naive definition and present a formal definition which is compatible with our naive definition.

Definition 1.1. A function f from a set X to a set Y is a relation between the elements of X (called the inputs) and the elements of Y (called the outputs) with the property that each input is related to **one and only one** output. We use the notation

$$f: X \rightarrow Y$$

to denote a function from X to Y and we call X the **domain** of f and Y the **codomain** of f .

The following is standard terminology associated to a function which is used to describe the outputs of a function.

Terminology 1.2. Given an input $x \in X$ of a function $f: X \rightarrow Y$, by definition, there is a unique output element $y \in Y$ related to x . We say that “ f sends x to y ” and write $x \mapsto y$ or $f: x \mapsto y$. The element y is usually denoted by $f(x)$ and is called:

- f of x
- the output of f for the input x
- the value of f at x
- the image of x under f

The set of all values of f is called the **range** of f , or the **image** of X under f . Symbolically,

$$\text{range of } f = \text{image of } X \text{ under } f = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

The following is standard terminology associated to a function which is used to describe the inputs which are related to a given output of a function.

Terminology 1.3. Given an element $y \in Y$, an element $x \in X$ with $f(x) = y$ is called a **preimage** of y or an **inverse image** of y . The set of all inverse images of y is called **the inverse image of y** . Symbolically,

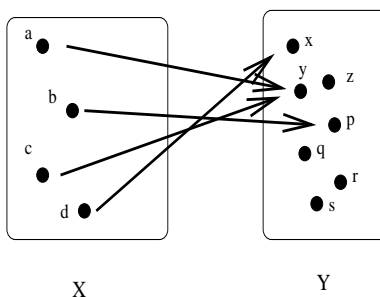
$$\text{inverse image of } y = \{x \in X \mid f(x) = y\}$$

2. REPRESENTATIONS OF FUNCTIONS

There are many different ways to represent functions. We shall consider some of the more common representations here.

2.1. Arrow Diagrams. A very primitive way to represent a function is with an arrow diagram. Specifically, we sketch all elements in the domain on the left side of the page, all elements from the codomain on the right hand side and then sketch arrows from each domain element to its image. We illustrate.

Example 2.1. Answer the following questions about the function $f: X \rightarrow Y$ defined by the following arrow diagram.



(i) Determine the domain and the codomain

The domain is $\{a, b, c, d\}$ and the codomain is $\{x, y, z, p, q, r, s\}$.

(ii) What is the range of f ?

The range is all elements in the codomain who are in the image of f , so $\{x, y, p\}$.

(iii) What is the inverse image of y ?

The inverse image of y is the set of all points in X whose image is y , so $\{a, c\}$.

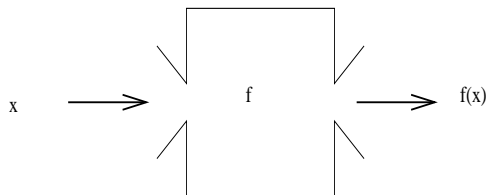
2.2. Functions as Ordered Pairs. Functions can also be thought of as a set of ordered pairs. Specifically, we can represent a function as a set of ordered pairs where the first coordinate runs over the domain X , and the second coordinate is the corresponding image in Y . This representation has obvious graphical interpretations. We illustrate.

Example 2.2. Represent the function given by the arrow diagram in Example 2.1 as a set of ordered pairs.

We have

$$f := \{(a, y), (b, p), (c, y), (d, x)\}$$

2.3. Functions as Machines. In elementary algebra, functions are often thought of as machines, where at one end of the machine you insert elements from the domain, and at the other end of the machine comes elements from the codomain.



The purpose of thinking of a function in this way is that it de-emphasises numerical formulas which are normally associated with functions (which is important since functions can and are defined on many different types of sets - not just numbers). We illustrate.

Example 2.3. We can define a function as follows: the input is a list of true/false questions and the outputs are the answers given by a person answering all questions. Since the person answers all questions as either true or false, this defines a function. The arrow diagram would simply connect all questions which were answered “true” with the word “true” and all questions which were answered “false” with the word “false”.

Example 2.4. Is the color of a car a function of its make? Why or why not?

If the color of a car is a function of make, that would mean when we input a make into the function, it would output a color. The only way this could be a function is if the car manufacturers made only a single color, which is definitely not the case. Thus this is not a function.

2.4. Functions Defined by Formulas. The traditional way to represent a function whose domain and range are sets of numbers is through a formula. Specifically, if x is representative of an arbitrary element from the domain of f , then a formula representation of f would be an algebraic equation in x whose values are equal to the range values whenever a specific value is entered into the equation. When such a representation is given the domain is usually assumed to be the subset of real numbers for which the equation is defined (unless otherwise stated). We illustrate.

Example 2.5. Consider the functions f defined by $f(x) = 2$. This is called the constant function with value 2 since the value of f at any x is always 2. Since the domain is not stated, we would normally assume it is \mathbb{R} since it is defined for every $x \in \mathbb{R}$.

Warning. To avoid confusion with the value of a function and a representation of a function, we shall use f to represent a function and

$f(x)$ to represent the value of the function at x (not to represent the function itself - this is in stark contrast to most calculus and algebra courses).

3. CLASSIC EXAMPLES AND PROBLEMS

Now we have formally defined a function, stated all the terminology and provided some representations, we shall consider some explicit examples. Before we do this however, we shall define what it means for two functions to be equal.

Definition 3.1. Suppose f and g are functions from X to Y . Then f equals g written $f = g$ if and only if

$$f(x) = g(x), \forall x \in X$$

Note that this implies two things:

- Both f and g have the same domain
- The image of every point in X is the same under both g and f .

Example 3.2. Consider the functions $f(x) = \sqrt{x^2}$ and $g(x) = x$. Is $f = g$?

In this case the domains are the same. However, the ranges are not. Specifically, for and $x < 0$, we have $f(x) = -x$ and $g(x) = x$ i.e. $f(-1) = \sqrt{(-1)^2} = 1$ and $g(-1) = -1$. In particular, these are not equal, so $f \neq g$.

Example 3.3. Consider the functions $f(x) = (\sqrt{x})^2$ and $g(x) = x$. Is $f = g$?

In this case, for all $x \geq 0$, $f(x) = g(x)$. However, for $x < 0$, f is not defined and g is. In particular, these are not equal, so $f \neq g$.

Example 3.4. Suppose X is any set. Then we define the identity function i_X on X to be $i_X(x) = x$ for all $x \in X$. The domain and range of this function is the set X .

Example 3.5. Suppose $a_1, a_2 \dots$ is a sequence of real numbers. We can define a function $f: \mathbb{Z}^* \rightarrow \mathbb{R}$ by $f(n) = a_n$ i.e. we send the index of a term in the sequence to its value in the sequence. This is an important way to regard sequences since it allows us to develop many of the convergence/divergence tests for the corresponding series.

Example 3.6. Let X be a set and $\mathcal{P}(\mathcal{P}(X))$ be its power set. We define a function $f: X \rightarrow \mathcal{P}(\mathcal{P}(X))$ as $f(x) =$ "the set of subsets of X which contain x ". If $X = \{x, y, z\}$, calculate $f(x)$.

We have

$$f(x) = \{\{x\}, \{x, y\}, \{x, z\}, \{x, y, z\}\}$$

Logarithms are important functions in mathematics. They are defined as follows.

Definition 3.7. Let b be a positive real number with $b \neq 0$. For each positive real number x , we define the logarithm base b of x to be the number y such that $b^y = x$. Symbolically,

$$\log_b(x) = y \iff x = b^y.$$

Example 3.8. Calculate the following:

(i) $\log_2(8)$

We have $\log_2(8) = 3$ since $2^3 = 8$.

(ii) $\log_5(1/25)$

We have $\log_5(1/25) = -2$ since $5^{-2} = 1/25$.

(iii) $\log_3(3^n)$ for any n

We have $\log_3(3^n) = n$ since $3^n = 3^n$ for any n .

(iv) $\log_a(1)$ for any $a > 0$

We have $\log_a(1) = 0$ since $a^0 = 1$ for any $a > 0$.

4. THE NOTION OF “WELL DEFINED”

In some cases, it may be apparent that a relationship between two quantities exists, but this relationship is not actually a function. The most common situation where a given relationship is not modeled by a function is when there is more than one “output” for some given “input”. Under such circumstances, we say that the relationship is **not well defined as a function**. We illustrate.

Example 4.1. Which of the following relationships are not well defined functions.

(i) $f: \mathcal{P}(X) \rightarrow X$ defined by $f(S) =$ “the first element in S ” where X is any set.

This is not a well defined function since there is no notion of “first element in a set” i.e. the set $S = \{x, y\}$ is the same as the set $S = \{y, x\}$, but these two different representations would give different values for f .

(ii) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) =$ “the closest integer to x ”.

This is not a well defined function since there is no single closest integer to $1/2$ (or indeed any integer plus $1/2$ i.e. $f(1/2) = 1$ or $f(1/2) = 0$ and it is impossible to determine which.

Homework

(i) From the book, pages 399-402: Questions: 2, 3, 5a, 6a, 7, 10, 11, 13, 16, 17, 21, 27, 31, 34, 35a, 35e, 42, 44