Section 7.3: Application: The Pigeon Hole Principle

In this section, we shall consider an application of the results we have developed. Though the result we shall consider is obviously true, we shall see that it can be used to prove a number of different things which are themselves not so obvious.

1. The Pigeon Hole Principle

We shall first state and prove the pigeon hole principle.

Theorem 1.1. *Suppose X and Y are finite sets and $N(X) > N(Y)$. Then a function $f\colon X \to Y$ cannot be one-to-one.*

Proof. Suppose that $N(X) = m$ and $N(Y) = n$ and $n < m$. We shall prove this result by contradiction, so assume that $f\colon X \to Y$ is a one-to-one function. We shall consider the inverse image, $f^{-1}(y)$ of each y in the codomain of f.
First observe that for $y_1 \neq y_2$ in the codomain of f, we have $f^{-1}(y_1) \cap f^{-1}(y_2) = \phi$ (since by definition of a function, an element of X cannot map under f to two elements of Y). It follows if $y_1, \ldots, y_r$ are the elements in the codomain of f (so $r \leqslant n$), then

$$N(X) = N(f^{-1}(y_1)) + N(f^{-1}(y_2)) + N(f^{-1}(y_3)) + \ldots N(f^{-1}(y_r))$$

Next note that if f is one-to-one, then $N(f^{-1}(y_i)) = 1$ for all i, and thus

$$m = N(X)$$
$$= N(f^{-1}(y_1)) + N(f^{-1}(y_2)) + N(f^{-1}(y_3)) + \ldots N(f^{-1}(y_r)) = r \leqslant n$$

In particular, $m \leqslant n$, which contradicts our initial assumption. Thus there cannot exists a one-to-one function $f\colon X \to Y$.

□

We consider some examples of how to use the pigeon hole principle.

Example 1.2. Prove that there must be at least two people in Portland with the same number of hairs on their heads.

A typical head of hair has around $150,000$ hairs, so it is reasonable to assume that no one has more than $1,000,000$ hairs on their head. The population of the Portland metropolitan area is about $2,000,000$ people. Let X be the set of all Portland residents and define a function $f\colon X \to \mathbb{Z}$ by $f(x) =$"the number of hairs on x's head". Note that since no one has more than $1,000,000$ hairs on their head, the codomain will consist of the numbers 0 through $1,000,000$, so will be of total size $1,000,001$. The size of X is at least 2 million people. In particular, the size of the domain is larger than the size of the codomain, and thus by the pigeon hole principle, the function f cannot be one-to-one. In particular, there exists at least two people x_1 and x_2 with $f(x_1) = f(x_2)$ i.e. they have the same number of hairs on their heads.

Example 1.3. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and suppose that 6 integers are chosen from S. Must there be two integers whose sum is 10? What if 5 integers are chosen?

We shall try to use the pigeon hole principle to show that there must be two integers whose sum is 10. To do this, we need to set up a function between finite sets. Let

$$X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$$

be the set of six integers we choose from S and let

$$Y = \{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}\},$$

a partition of S. We define $P\colon X \to Y$ as $P(a_i) =$"the set containing a_i". Note that $N(X) = 6$ and $N(Y) = 5$ and therefore, P cannot be a one-to-one function. In particular, there must be an element of Y which contains two images from elements in X i.e. there must be at least one set of $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}$ which contains the image of two elements of X. However, each of these sets contain two integers whose sum is 10, and thus the image of P must contain two integers whose sum is 10.
Alternatively, if we choose 5 integers, we can choose $1, 2, 3, 4, 5$, and we observe that the sum of any two of these integers will not equal 10. Note that the argument breaks down in this case since we would have $N(X) = 5$ and $N(Y) = 5$, so we could not use the pigeon hole principle.

Example 1.4. Show that in a room with several people, there are at least two people with the same number of acquaintances in that room (acquaintance is a two way relationship).

Suppose that there are N people in a room. Let X be the set of people and define $f\colon X \to \mathbb{Z}$ be defined by $f(x) =$"the number of people x is acquaintance with". The most number of people that anyone can be acquaintanced with is $n - 1$ and the least is 0. Therefore, the codomain will be $Y = \{0, 1, 2, 3 \ldots, n - 1\}$. Now we cannot apply the pigeon hole principle since $N(X) = N(Y)$. However, observe that it is not possible that it is not possible that there exists $x_1, x_2 \in X$ with $f(x_1) = 0$ and $f(x_2) = n - 1$. In particular, the codomain will either be $Y = \{1, 2, 3, \ldots, n - 1\}$ or $Y = \{0, 1, 2 \ldots, n - 1\}$. In particular, in both cases, $n = N(X) > N(Y) = n - 1$, and so the pigeon hole principle guarantees that f is not one-to-one i.e. there is at least two people with the same number of acquaintances.

Example 1.5. How many integers must you pick in order for them to have the same remainder when divided by 7?

Let $X = \{a_1, \ldots, a_n\}$ be the set of integers you have chosen and let $Y = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of possible remainders after division of a number by 7. We can define a function $f\colon X \to Y$ as $f(x) =$"remainder after division by 7". Since $|X| = n$, and $|Y| = 7$, if we choose $n \geqslant 8$, the pigeon hole principle guarantees that f will not be one-to-one and in particular, that there will be at least two integers whose remainders after dividing by 7 are equal.

2. The Generalized Pigeon Hole Principle

In fact, there is are other more general statements related to the pigeon hole principles whose validity arises from the same principles as the pigeon hole principle. We shall not prove them since the proofs are similar to the regular pigeon hole principle.

Theorem 2.1. *(The Generalized Pigeon Hole Principle) Suppose X and Y are finite sets and $N(X) > k \cdot N(Y)$ where k is a positive integer. Then if $f\colon X \to Y$ is a function, there exists some y in the codomain of f such that $f^{-1}(y)$ has at least $k+1$ distinct elements i.e. there is some $y \in Y$ which is the image of at least $k+1$ elements from X.*

A useful related statement is the contrapositive.

Theorem 2.2. *(The Generalized Pigeon Hole Principle Contrapositive) Suppose X and Y are finite sets. If $f\colon X \to Y$ is a function and $N(f^{-1}(y)) \leqslant k$ for all $y \in Y$, then $|X| \leqslant k \cdot |Y|$.*

We illustrate with some examples.

Example 2.3. In a group of 1500 people, must at least five people have the same birthday?

Let X be a set of $1,500$ people, let $Y =$"the set of all possible birthdays" and define $f\colon X \to Y$ by $f(x) =$"x's birthday". Since

$$|X| = 1500 > 1460 = 4 \cdot 365 = 4 \cdot |Y|$$

it follows from the generalized pigeon hole principle that there is at least one $y \in Y$ such that $f^{-1}(y)$ has at least 5 elements i.e. at least five people have the same birthday.

One important consequence of the pigeon hole principle is the following theorem for finite sets. We note that this result only holds for finite sets (as we shall see when we consider infinite sets in the next section).

Theorem 2.4. *Let X and Y be sets and suppose $N(X) = N(Y)$. Then a function $f\colon X \to Y$ is one-to-one if and only if it is onto.*

Proof. Suppose that $f\colon X \to Y$ is one-to-one and let C denote the codomain of f. Since f is one-to-one, it follows that if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. In particular, C will consist of $N(X)$ total elements. Since $C \subseteq Y$ and $N(C) = N(X) = N(Y)$, it follows that we must have $C = Y$ and hence f is onto.
Now suppose f is onto and let $Y = \{y_1, \ldots, y_n\}$. Since f is onto, for every $y \in Y$, there exists $x \in X$ with $f(x) = y$, so for a given $y_i \in Y$, let $x_i \in X$ be such that $f(x_i) = y_i$. Let $D = \{x_1, \ldots, x_n\}$. Then $D \subseteq X$, but since $N(D) = N(Y) = N(X)$, it follows that $X = D$. In particular, since each x_i can map to only one element in Y, it follows that for any $y \in Y$, there exists a unique $x \in X$ with $f(x) = y$ and hence f is one-to-one.

□

Homework

(i) From the book, pages 430-431: Questions: 2, 4, 6, 7, 11, 12, 13, 18, 24, 25, 26, 30, 35