

## Section 7.5: Cardinality

In this section, we shall consider extend some of the ideas we have developed to infinite sets. One interesting consequence of this discussion is that we shall see there are as many integers as there are rational numbers - a very bizarre conclusion!

### 1. CARDINALITY

We want to define the notion of when two sets with infinitely many elements have the same size. In order to do this, we shall generalize the ideas we developed for finite sets. Recall that a one-to-one correspondence between a set  $X$  and a set  $Y$  is a function  $f: X \rightarrow Y$  which is one-to-one and onto. If  $X$  and  $Y$  are finite sets, and there exists a one-to-one correspondence between  $X$  and  $Y$ , then clearly  $X$  and  $Y$  have the same size. This motivates the following definition.

**Definition 1.1.** Let  $X$  and  $Y$  be sets. We say that  $X$  has the same cardinality as  $Y$  if and only if there is a one-to-one correspondence from  $X$  to  $Y$  (when considering general sets, we use the term “cardinality” instead of “size”).

The following results are clear regarding cardinality.

**Theorem 1.2.** *Suppose  $X$ ,  $Y$  and  $Z$  are sets.*

- (i) *(Reflexive property)  $X$  has the same cardinality as  $X$*
- (ii) *(Symmetric Property) If  $X$  has the same cardinality as  $Y$ , then  $Y$  has the same cardinality as  $X$*
- (iii) *(Transitive Property) If  $X$  has the same cardinality as  $Y$  and  $Y$  has the same cardinality as  $Z$ , then  $X$  has the same cardinality as  $Z$*

*Proof.* The proofs are all straight forward, we simply need to build a one-to-one correspondence between the stated sets..

- (i) The identity map  $i_X: X \rightarrow X$  is a one-to-one correspondence from  $X$  to itself.
- (ii) If  $X$  has the same cardinality as  $Y$ , then there exists a one-to-one correspondence  $f: X \rightarrow Y$ . Then the map  $f^{-1}: Y \rightarrow X$  will be a one-to-one correspondence, and thus  $Y$  has the same cardinality as  $X$ .
- (iii) If  $X$  has the same cardinality as  $Y$ , and  $Y$  has the same cardinality as  $Z$ , then there exists one-to-one correspondences  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ . Then the map  $g \circ f: X \rightarrow Z$  will be a one-to-one correspondence, and thus  $X$  has the same cardinality as  $Z$ .

□



We define  $f(1) = 1$ , and in general  $f(n)$  to be the rational number in the table obtained after following  $n - 1$  arrows given in the diagram (note that the arrows skip over terms which have already been repeated). By construction, this function will be one-to-one (since we are skipping all repeated terms). It is onto, since every positive rational number will appear in the table. It follows that  $f$  is a one-to-one correspondence, and thus  $\mathbb{Q}$  is infinitely countable.  $\square$

In addition to proving that other sets we know are countable, we can also consider more general results which hold for countable sets, such as the following.

**Theorem 2.4.** *Any subset of a countable set is countable.*

*Proof.* Suppose  $A$  is a countable set and suppose  $B \subseteq A$ . If  $B$  is finite, then by definition it is countable and hence the result holds. Therefore suppose that  $B$  is infinite.

Since  $A$  is countable, there exists a one-to-one correspondence  $f: \mathbb{Z}^+ \rightarrow A$ . We need to construct a one-to-one correspondence  $g: \mathbb{Z}^+ \rightarrow B$ . We do this as follows. Since  $A$  is countable, we can write out the elements of  $A$  in order according to the function  $f$  i.e.  $a_1, a_2, a_3, \dots$  where  $a_i = f(i)$ . Since all the elements of  $B$  are in  $A$ , they will each appear somewhere in this list. For  $n \in \mathbb{Z}$ , we define  $f(n) =$ “the  $n$ th element of  $B$  which appears in the list starting from  $a_1$ ”. By construction, this is one-to-one. It is onto since every element of  $B$  is in  $A$ , so appears somewhere in this list. Hence  $g: \mathbb{Z}^+ \rightarrow B$  is a one-to-one correspondence, so  $A$  and  $B$  have the same cardinality, and hence  $B$  is countable.  $\square$

### 3. UNCOUNTABLE SETS

After our observations in the previous section, we saw that many infinite sets we are familiar with are countable infinite. This leads to the obvious question of whether there are any sets of number which are infinite in size, but are uncountable. Our first goal is to show that there are other sets, and consequently, there are in fact more than one different size of infinity!

**Theorem 3.1.** *The set of real numbers between 0 and 1 is uncountable.*

*Proof.* We shall prove this by contradiction. Suppose that the set of numbers between 0 and 1 is uncountable. Then they can be written as

a list:

$$\begin{aligned} 1 &\mapsto a_1 = 0.a_{11}a_{12}a_{13}a_{14}\dots \\ 2 &\mapsto a_2 = 0.a_{21}a_{22}a_{23}a_{24}\dots \\ 3 &\mapsto a_3 = 0.a_{31}a_{32}a_{33}a_{34}\dots \\ 4 &\mapsto a_4 = 0.a_{41}a_{42}a_{43}a_{44}\dots \\ 5 &\mapsto a_5 = 0.a_{51}a_{52}a_{53}a_{54}\dots \\ &\vdots \end{aligned}$$

i.e we can list them according to the function  $f: \mathbb{Z}^+ \rightarrow (0, 1)$  where  $a_{ij}$  is the  $j$ th decimal place of the  $i$ th number. To derive a contradiction, we shall construct a number  $b$  between 0 and 1 which does not appear in this list. We define

$$b = 0.b_1b_2b_3b_4 \text{ where } b_i = \begin{cases} 3 & a_{ii} \neq 3 \\ 4 & a_{ii} = 3 \end{cases}.$$

By construction,  $b$  is a number between 0 and 1. Also, it cannot possibly lie in the list constructed above. Specifically, if  $b$  appears in the list, then there is some number  $a_i$  such that  $b = a_i$ . It follows that every decimal place of  $a_i$  and  $b$  should match. However, the  $i$ th decimal place of  $a_i$  will be  $a_{ii}$ , and the  $i$ th decimal place of  $b$  will be 3 if  $a_{ii} \neq 3$  and 4 if  $a_{ii} = 3$  - in either case,  $b_i \neq a_{ii}$  and hence  $b \neq a_i$ . This contradicts our assumption that this set is countable, and hence the set of real numbers from 0 to 1 is uncountable.  $\square$

This proves that there are sets which are uncountable. In fact, many of the other sets we know which are not countable have the same cardinality as the set of numbers between 0 and 1. We illustrate.

**Example 3.2.** The set of real numbers has the same cardinality as the set of numbers between 0 and 1.

We define a function

$$f: (0, 1) \rightarrow \mathbb{R} \text{ by } f(x) = -\frac{x - \frac{1}{2}}{x(x - 1)}$$

Notice that  $f(x)$  is always increasing since

$$f'(x) = \frac{x^2 + \frac{1}{2}}{x^2(x - 1)^2} > 0.$$

It is also defined for all  $x \in (0, 1)$ , and thus it must be one-to-one in the interval  $(0, 1)$ . Next note that

$$\lim_{x \rightarrow 0^+} f(x) = -\infty \text{ and } \lim_{x \rightarrow 1^-} f(x) = \infty$$

and thus  $f$  must be onto. Thus  $f$  is a bijection, and so  $(0, 1)$  and  $\mathbb{R}$  have the same cardinality.

We have the following nice result regarding uncountable sets which is a corollary to Theorem 2.4.

**Corollary 3.3.** *Any set with an uncountable subset is uncountable.*

*Proof.* Theorem 2.4 states that any subset of a countable set is countable. Putting this in logical form, for  $S \subseteq X$ , it states that if  $X$  is countable then  $S$  is countable. This is equivalent to the statement that if  $S$  is uncountable, then  $X$  is uncountable, which is exactly the statement we are trying to prove. □

#### 4. THE CONTINUUM HYPOTHESIS

Our observations lead naturally on to a number of follow up questions. For example:

- (i) Is there more than two infinities?
- (ii) Is there a set whose cardinality is smaller than  $\mathbb{R}$  but larger than  $\mathbb{Z}^+$ ?

These questions are part of a more general theory known as the study of transfinite numbers (they are all transfinite numbers). Under certain set theoretical axioms, it can be shown that there are in fact infinitely many different infinities, and hence the answer to the first question is a resounding yes. The second question however is much deeper and is usually referred to as “The Continuum Hypothesis”. This is one of the famous problems in mathematics posed by David Hilbert in 1900 (one of the original millenium problems), and was supposedly proved to be independent of the axioms of set theory (i.e. not provable or provable) by Paul Cohen in 1963 (though his result is not accepted by everyone). Currently it is still a source of major research in mathematics.

#### Homework

- (i) From the book, pages 454-456: Questions: 2, 5, 10, 16, 20, 21, 25, 32, 37