# Research Statement

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## 1 Primary Research Results

Historically, compact Riemann surfaces were constructed as the domains of definition of multivalued functions. Specifically, given an irreducible equation $\Phi(z, w) = 0$ of two variables, using analytic continuation, one can construct a compact domain on which one variable of the equation $\Phi$ is a single valued function of the other. Conversely, all compact Riemann surfaces can be realized in such a way (for details see [5]).

Presently there is no known method to write down a defining equation given an arbitrary abstract compact Riemann surface $X$. In addition, given two different equations, there is no general method to determine if the underlying compact Riemann surfaces are conformally equivalent. This suggests that an open problem in the theory of compact Riemann surfaces is to find a unique defining equation given any such surface up to conformal equivalence. Except for a small number of cases, for a fixed genus, such a task is unreasonable. A more common problem is to focus on certain families of surfaces with symmetry (see [4] and [6]).

A compact Riemann surface $X$ is said to be a cyclic $p$-gonal surface if it admits an automorphism $\sigma$ of prime order $p$ such that the quotient space $X/\langle\sigma\rangle$ has genus 0. The group $\langle\sigma\rangle$ is called a $p$-gonal group for $X$. It is said to be normal cyclic $p$-gonal if in addition the group $\langle\sigma\rangle$ is normal in the full automorphism group of $X$. The primary result from my dissertation is as follows:

- There is a method for producing **explicit** defining polynomial equations for cyclic $p$-gonal surfaces for a fixed prime $p$. In the case of normal cyclic $p$-gonal surfaces, this method can be extended so that the equations are unique for each surface up to conformal equivalence (for details see [8] which can be accessed via my website).

Following this, my main focus has been to improve upon these results. The main objective is to complete the following project: for a fixed genus $g$, write down a list of unique equations for all surfaces which admit a cyclic prime cover of the Riemann sphere for *any* prime $p$. This question motivates the following two problems:

1. If $X$ is not normal $p$-gonal, what, if anything, can be said about the uniqueness of equations constructed using [8]?

2. Could $X$ admit cyclic prime covers of the Riemann sphere for different primes? If so, can we classify all such surfaces?

To answer the first question, it was noticed that the results of [8] are a little more general than the author first realized. Specifically, the results hold provided there exists a unique conjugacy class of $p$-gonal groups in the full automorphism group of $X$. The following result, which is interesting in its own right, was proved by the author in [9].

**Theorem** *If $X$ is cyclic $p$-gonal, then there exists a unique conjugacy class of $p$-gonal groups for $X$ in $\mathrm{Aut}(X)$, the full automorphism group of $X$.*

Our previous remarks and this result imply that the equations constructed in [8] are unique for a fixed prime $p$ for all $p$-gonal surfaces. This answers the first problem.

The second problem tackles the issue that for a fixed genus, though we can find unique defining equations for $p$-gonal surfaces for a fixed prime $p$, a surface may admit cyclic prime covers of the Riemann sphere for different primes. In [8], the equations constructed depend upon the prime $p$. This means that in a list of equations for surfaces which admit a cyclic prime cover of the Riemann sphere of a fixed genus, different equations will be equations for the same surface depending on the prime used to construct the equation. This is still an active area of research. The following result has been proved by the author in [10].

**Theorem** *A surface can admit cyclic prime covers of the Riemann sphere for at most two different primes.*

Using this result, we only need examine surfaces which admit cyclic prime covers of the Riemann sphere for two distinct primes $p$ and $q$.

Suppose that $X$ is a cyclic $p$-gonal and a cyclic $q$-gonal surface. Let $C_p$ and $C_q$ denote a $p$-gonal and a $q$-gonal group for $X$ respectively and let $G$ denote the group generated by $C_p$ and $C_q$. It is conjectured by the author that $C_p$ and $C_q$ can be chosen so that the group $G$ has order $pq$. Calculations for small genus and small primes imply this conjecture, though as of yet a complete proof has not been developed. Under assumption of this conjecture, in [10], the author classifies all surfaces which are both cyclic $p$-gonal and cyclic $q$-gonal for primes $p \neq q$.

# 2 Future Research Projects

The research I have pursued on cyclic $p$-gonal surfaces is still an ongoing project. Presently I am working on a proof of the conjecture stated above.

Following completion of this project, there are many natural extensions or generalizations to consider. Similar methods could be used to find equations for generalized $p$-gonal surfaces - surfaces which admit a cyclic prime cover of a surface of genus $g > 0$. A. Weaver, [7] has studied some of the properties of generalized hyperelliptic curves. B. Estrada and E. Martinez in [1] have studied some of the properties of trigonal surfaces (where $p = 3$). These results together with structure theorems of Fuchsian groups could yield a way to find defining equations for such surfaces. If a method could be developed for $p = 2$ and 3, then it may be possible to generalize it for all $p$.

Aside from these projects, for some time I have been considering a generalization of Belyĭ's theorem. The generalization would be that a compact Riemann surface $X$ is defined over a number field of transcendence degree $n$ if and only if there exists a function ramified over $3 + n$ points and no function ramified over fewer points. The research I have pursued on cyclic prime Galois covers of the sphere ramified over $n$ points together with a study of the structure of Fuchsian groups should provide some insight into this generalization. In fact, it is my belief and intention to use Fuchsian groups to prove this claim.

Also of interest is how information about groups of automorphisms of compact Riemann surfaces of a fixed genus $g$ may yield knowledge regarding the mapping class group of genus $g$. For the genus 1 case, the mapping class group is isomorphic to the free product of a cyclic 2 and a cyclic 3 group. These groups are the only groups to appear as finite subgroups of the full automorphism groups for genus 1 in addition to the generic full automorphism group in genus 1. It was shown in [3] that for genus $g \geq 2$, any finite subgroup of the mapping class group in genus $g$ is necessarily a group of automorphisms of a compact Riemann surface of that genus. More recently, it has been shown that the mapping class group is generated by elements of finite order, see [2]. These results imply that a thorough analysis of the finite subgroups of the mapping class group of genus $g$ may provide insight in a search for a presentation of this group. It is my intention to use the recent progress in the search for automorphism groups of compact Riemann surfaces to perform such an analysis for low genus and this will hopefully provide information which may be used to draw conclusions for arbitrary genus.

# References

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