Section 2.6: Limits at Infinity and Horizontal Asymptotes

In previous sections we considered limits at finite points. In addition to a discussion on finite limits, we extended the definition to also include infinite limits (or vertical asymptotes). In this section we also consider infinite limits, but in this case we are considering the limit as x grows in size without bound toward ∞ or $-\infty$.

1. The Definition of Continuity

We start with some definitions of infinite limits.

Definition 1.1. Suppose that $f(x)$ is some function and L is a number.

(*i*) If $f(x)$ is defined on some interval $(-\infty, a)$, then we say the limit of $f(x)$ as x goes to $-\infty$ is L and write

$$\lim_{x\to-\infty} f(x) = L$$

if the values of $f(x)$ get closer and closer to L as $x \to -\infty$.

(*ii*) If $f(x)$ is defined on some interval (a, ∞), then we say the limit of $f(x)$ as x goes to ∞ is L and write

$$\lim_{x\to\infty} f(x) = L$$

if the values of $f(x)$ get closer and closer to L as $x \to \infty$.

If either

$$\lim_{x\to\infty} f(x) = L$$

or

$$\lim_{x\to-\infty} f(x) = L$$

we call the line $y = L$ a horizontal asymptote of the function $f(x)$.

We illustrate with some explicit examples.

Example 1.2. (*i*) Determine the horizontal asymptotes of $f(x) = e^{-x}$.

Observe that as $x \to \infty$, we have $f(x) \to 0$. Thus the line $y = 0$ is a horizontal asymptote for $f(x)$.

(*ii*) Determine the limits at infinity of $f(x) = \arctan(x)$

Since $\arctan(x)$ is the inverse of $\tan(x)$ on $(-\pi/2, \pi/2)$, and we know that $\lim_{x\to\pi/2} \tan(x) = \infty$ and $\lim_{x\to-\pi/2} \tan(x) = -\infty$, it follows that

$$\lim_{x\to\infty} \arctan(x) = \pi/2$$

and

$$\lim_{x\to-\infty} \tan(x) = -\pi/2.$$

(*iii*) Determine the limits at infinity of

$$R(x) = \frac{2x^2 - 2x + 1}{4x - 5x^2 + 2}.$$

Observe that this is a rational function, so the end behavior is determined by the quotients of the leading terms. In this case, we have

$$\lim_{x\to\infty} \frac{2x^2 - 2x + 1}{4x - 5x^2 + 2} = \lim_{x\to\infty} \frac{2x^2}{-5x^2} = -\frac{2}{5}.$$

Likewise, we have

$$\lim_{x\to-\infty} \frac{2x^2 - 2x + 1}{4x - 5x^2 + 2} = -\frac{2}{5}.$$

(*iv*) Determine the infinite limit

$$\lim_{x\to\infty} (\sqrt{(9x^2 - x)} - 3x).$$

This limit looks like it could be either undefined or 0. However, using simple algebra, we shall show that it is neither of these. We have

$$\lim_{x\to\infty} (\sqrt{(9x^2 - x)} - 3x) \lim_{x\to\infty} (\sqrt{(9x^2 - x)} - 3x) \frac{\sqrt{(9x^2 - x)} + 3x}{\sqrt{(9x^2 - x)} + 3x}$$

$$= \lim_{x\to\infty} \frac{9x^2 - x - 9x^2}{\sqrt{(9x^2 - x)} + 3x} = \lim_{x\to\infty} \frac{-x}{\sqrt{(9x^2 - x)} + 3x}$$

$$= \lim_{x\to\infty} \frac{-x}{\sqrt{(9x^2 - x)} + 3x} \left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) = lim_{x\to\infty} \frac{-1}{\sqrt{(9 - \frac{1}{x})} + 3} = \frac{-1}{\sqrt{9} + 3} = -\frac{1}{6}$$

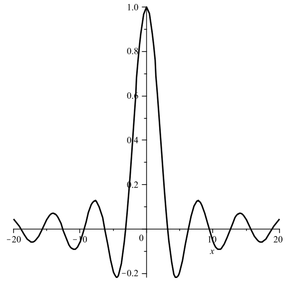
There are a number of interesting facts we can derive about horizontal asymptotes.

Example 1.3. (*i*) Can a function pass through a horizontal asymptote?

Yes - a function cannot pass through a vertical asymptote, but it can certainly pass through a horizontal asymptote. Take for example

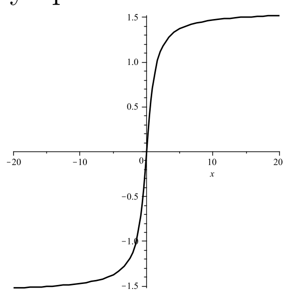
$$f(x) = \frac{\sin(x)}{x}$$

which has horizontal asymptote $y = 0$, but it passes through it infinitely many times.



(*ii*) If a function has a horizontal asymptote, will it be the same on both sides?

A horizontal asymptote does not have to be the same on both sides. One specific example of a function with two different horizontal asymptotes is the inverse tangent function:



2. Infinite Limits at Infinity

Just as when we are taking a limit at a finite we can have an infinite limit, it can also happen when we are considering limits at ∞. Specifically we have the following:

Definition 2.1. Suppose that $f(x)$ is some function.

(*i*) We write
$$\lim_{x\to-\infty} f(x) = \infty$$
to mean as $x \to -\infty$, $f(x) \to \infty$.

(*ii*) We write
$$\lim_{x\to-\infty} f(x) = -\infty$$
to mean as $x \to -\infty$, $f(x) \to -\infty$.

(*iii*) We write
$$\lim_{x\to\infty} f(x) = \infty$$
to mean as $x \to -\infty$, $f(x) \to \infty$.

(*iv*) We write
$$\lim_{x\to\infty} f(x) = -\infty$$
to mean as $x \to -\infty$, $f(x) \to -\infty$.

We illustrate with some examples.

Example 2.2. Evaluate the following limits.

(i)

$$\lim_{x\to\infty} x^2 - x^6 + 2x$$

This is a polynomial, so the end behavior is the same as that of the leading term, $-x^6$. Thus we have

$$\lim_{x\to\infty} x^2 - x^6 + 2x = -\infty$$

(ii)

$$\lim_{x\to-\infty} \frac{x(x-2)}{x+2}$$

This is a rational function, so the end behavior is the same as that of the quotient of the leading term, x. Thus we have

$$\lim_{x\to\infty} \frac{x(x-2)}{x+2} = -\infty$$

(iii)

$$\lim_{x\to-\infty} e^{-x}\sin(x)$$

In this case $\sin(x)$ infinitely oscillates and e^{-x} gets larger and larger as $x \to \infty$. Thus the function $e^{-x}\sin(x)$ oscillates between larger and larger numbers as $x \to \infty$ and in particular, the limit cannot possibly exist.

(iv)

$$\lim_{x\to\infty} \frac{x-9}{\sqrt{(4x^2+3x+2)}}$$

Though this is not a polynomial or rational function, we can use the same idea to evaluate the end behavior. Specifically, we only consider the largest of the terms as $x \to \infty$ since all the other terms will contribute much less to the value relative to these terms. Specifically, we have

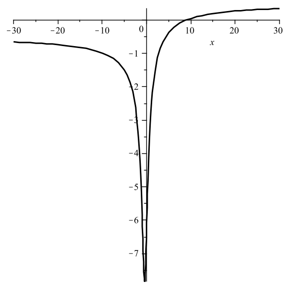
$$\lim_{x\to\infty} \frac{x-9}{\sqrt{(4x^2+3x+2)}} = \lim_{x\to\infty} \frac{x}{\sqrt{(4x^2)}} = \frac{1}{2}$$

(v)

$$\lim_{x\to-\infty} \frac{x-9}{\sqrt{(4x^2+3x+2)}}$$

We can use the same idea as the previous example. Specifically, we only consider the largest of the terms as $x \to -\infty$ since all the other terms will contribute much less to the value relative to these terms. Specifically, we have

$$\lim_{x\to-\infty} \frac{x-9}{\sqrt{(4x^2+3x+2)}} = \lim_{x\to-\infty} \frac{x}{\sqrt{(4x^2)}} = -\frac{1}{2}$$



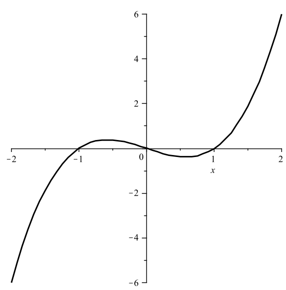
Example 2.3. Use your knowledge of limits at ∞ and otherwise to sketch a rough graph of $p(x) = x^3 - x$.
We know that

$$\lim_{x\to\infty} p(x) = \infty$$

and

$$\lim_{x\to-\infty} p(x) = -\infty.$$

We also know that $p(0) = p(1) = p(-1) = 0$. Thus since it is continuous, the graph of $p(x)$ must be similar to the graph given below:



3. The Formal Definition of Limits at Infinity

Just as with regular limits, we can formalize the idea of a limits at ∞. Below we state the formal definition for a finite limit at ∞. To define infinite limits at ∞, we modify the definitions in the obvious way.

Definition 3.1. Suppose $f(x)$ is some function and a is some constant.

(i) If $f(x)$ is defined on $(-\infty, a)$ then

$$\lim_{x\to-\infty} f(x) = L$$

if for all $\varepsilon > 0$, there exists a number N such that $x < N$ implies $|f(x) - L| < \varepsilon$.

(ii) If $f(x)$ is defined on (a, ∞) then

$$\lim_{x\to\infty} f(x) = L$$

if for all $\varepsilon > 0$, there exists a number N such that $x > N$ implies $|f(x) - L| < \varepsilon$.

We illustrate with an example.

Definition 3.2. Show that

$$\lim_{x\to\infty} \frac{1}{x^2} = 0.$$

We need to show that for all $\varepsilon > 0$, there exists a number N such that $x > N$ implies $|1/x^2| < \varepsilon$. However $|1/x^2| < \varepsilon$ is only true if $|x^2| > \frac{1}{\varepsilon}$ or $|x| > \frac{1}{\varepsilon^{\frac{1}{2}}}$. Thus for any ε, if we choose $N = \frac{1}{\varepsilon^{\frac{1}{2}}}$, then we are guranteed that for all x with $|x| < N$, we have $|1/x^2| < \varepsilon$.