

Alternating Series

Nearly all of the results we have considered so far deal with series with strictly positive terms. In this section we shall consider certain types of series with positive and negative terms and whether or not they converge.

1. THE ALTERNATING SERIES TEST

The series we consider are defined as follows.

Definition 1.1. An alternating series is a series whose terms are alternately positive and negative.

We look at a couple of examples.

- Example 1.2.**
- (i) The series $(-1)^n$ is an alternating series - for each odd n it is negative and for each even n it is positive.
 - (ii) The series $\sum \cos(x)$ is not alternating - it does take positive and negative values, but it does not alternate between them.

Observe that any alternating series is of the form

$$\sum (-1)^n a_n$$

or

$$\sum (-1)^{n+1} a_n$$

where the a_n 's are all positive. For such series, there is a very simple test for convergence.

Result 1.3. An alternating series

$$\sum (-1)^n a_n$$

or

$$\sum (-1)^{n+1} a_n$$

where the a_n 's are all positive converges if both of the following conditions hold:

- (i) $a_{n+1} \leq a_n$ for all n
- (ii)

$$\lim_{n \rightarrow \infty} a_n = 0$$

Therefore, to check whether a series with positive and negative numbers converges, we need to check the following:

- (i) The series is alternating.
- (ii) The sequence of absolute values is decreasing.
- (iii) The sequence of absolute values is going to 0.

If these three things are satisfied, then the sequence converges. Of course, just because these things are not satisfied does not mean that the sequence does not converge, it just means that the alternating series test cannot be applied. We illustrate with a number of examples.

Example 1.4. Determine which of the following converge.

(i)

$$-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \dots$$

The n th term for this series appears to be

$$(-1)^n \frac{n}{n+2}.$$

The series is certainly alternating. However, it is not decreasing, so we cannot apply the alternating series test. Observe that

$$\lim_{n \rightarrow \infty} (-1)^n \frac{n}{n+2}$$

does not exist because

$$\lim_{n \rightarrow \infty} \frac{n}{n+2} = 1,$$

so by the n th term test, the series does not converge.

(ii)

$$\sum (-1)^{n-1} \frac{e^{1/n}}{n}$$

This series is certainly alternating, decreasing and the limit of

$$\frac{e^{1/n}}{n}$$

is 0, so the alternating series test tells us that this series must converge.

(iii)

$$\sum (-1)^{n-1} \frac{\ln(n)}{n}$$

This series is certainly alternating, decreasing and the limit of

$$\frac{\ln(n)}{n}$$

is 0, so the alternating series test tells us that this series must converge.

(iv)

$$\sum (-1)^n \sin\left(\frac{\pi}{n}\right)$$

This series looks like it is not alternating (because of the sin). However, since $0 \leq \pi/n \leq \pi$, the function $\sin(\pi/n)$ will always be positive. The first term is 0, the next term is -1 , so it does not decrease in the first two terms. Observe however that for $n \geq 2$, the sequence $\sin \frac{\pi}{n}$ does decrease, so we can apply the

alternating series test to this part of the series (remember, the first two terms will not affect whether it diverges or converges). The series is certainly alternating, decreasing for $n \geq 2$ and the limit of

$$\sin\left(\frac{\pi}{n}\right)$$

is 0, so the alternating series test tells us that the series

$$\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$$

must converge and hence so does the series

$$\sum (-1)^n \sin\left(\frac{\pi}{n}\right).$$