Section 17.9
The Divergence Theorem

"Turning a Flux Integral into a Triple Integral"

The last result we consider is a generalization of Green's Theorem - converting a flux integral over a closed surface into a triple integral over the interior of the surface.

1. The Divergence Theorem

To state the divergence theorem, we need the following definition which uses the ideas we built up in the section on triple integrals.

Definition 1.1. A solid E is called a simple solid region if it is one of the types (either Type 1, 2 or 3) given in Section 16.6.

Examples of a simple solid regions are spheres, ellipsoids, portions of cylinders etc. We can now state the divergence theorem.

Result 1.2. Let E be a simple solid region and let S be the boundary of E with outward orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region which contains E. Then

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_E \text{div}\vec{F} dV.$$

This Theorem tells us that any flux integral over a **closed** surface can be converted into a triple integral over the interior (provided it is defined on the interior), and since flux integrals in general are much more difficult than triple integrals, this gives us an easier way to calculate them. We illustrate with some examples.

Example 1.3. (i) Use the divergence theorem to evaluate

$$\int\int_S (e^x \sin(y)\vec{i} + e^x \cos(y)\vec{j} + yz^2\vec{k}) \cdot d\vec{S}$$

where S is the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Applying the divergence theorem, we have

$$\int\int_S (e^x \sin(y)\vec{i} + e^x \cos(y)\vec{j} + yz^2\vec{k}) \cdot d\vec{S} = \int_0^1\int_0^1\int_0^1 (e^x \sin(y) - e^x \sin(y) + 2yz) dxdydz$$

$$= \int_0^1\int_0^1\int_0^1 2yz dxdydz = \int_0^1 y^2 z\Big|_0^1 dz = \int_0^1 z dz = \frac{1}{2}$$

(*ii*) Suppose

$$\vec{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{i} + \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{j} + \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{k}$$

and S is the cylinder $y^2+z^2=4$ with $-2 \leqslant x \leqslant 2$. Show that the integral

$$\int\int_S \vec{F}\cdot d\vec{S}$$

is the same as the integral

$$\int\int_{S_1} \vec{F}\cdot d\vec{S}$$

where S_1 is the unit sphere centered at the origin oriented outward.

Observe that $\mathrm{div}\vec{F} = 0$ (if you are unsure, check). However, we cannot apply the divergence theorem since $\vec{F}$ is not defined at $(0,0,0)$. Observe however that if S_1 is the unit sphere centered at the origin, and S_2 is the closed surface which consists of S (oriented outward) and S_1 (oriented inward), we can apply the divergence theorem to get

$$\int\int_{S_2} \vec{F}\cdot d\vec{S} = \int\int\int_{Int(S_2)} 0dV = 0.$$

However,

$$\int\int_{S_2} \vec{F}\cdot d\vec{S} = \int\int_S \vec{F}\cdot d\vec{S} - \int\int_{S_1} \vec{F}\cdot d\vec{S} = 0$$

so

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_{S_1} \vec{F}\cdot d\vec{S}.$$

(*iii*) Use the divergence theorem to evaluate

$$\int\int_S (3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k})\cdot d\vec{S}$$

where S is the surface of the cylinder $y^2+z^2=1$ and the planes $x=-1$ and $x=2$.

We have $\mathrm{div}\vec{F} = 3y^2+3z^2$, so applying the divergence theorem, we have

$$\int\int_S (3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k})\cdot d\vec{S} = \int\int\int_R (3y^2+3z^2)dV.$$

Converting to cylindricals along the x-axis, we have $3y^2+3z^2 = r^2$, $0 \leqslant r \leqslant 1$, $0 \leqslant \vartheta \leqslant 2\pi$, and $-1 \leqslant x \leqslant 2$, so

$$\int\int\int_R 3y^2+3z^2dV = \int_{-1}^{2}\int_0^{2\pi}\int_0^1 3r^3drd\vartheta dx = 2\pi(3)(\frac{3}{4}) = \frac{9\pi}{2}$$

As with Green's Theorem, and Stokes Theorem, there are ways to apply the divergence theorem indirectly. We illustrate with some examples.

Example 1.4. Let S be the open cone $z = \sqrt{(x^2+y^2)}$ with $z \leqslant 3$. Calculate

$$\int\int_S \vec{F}\cdot d\vec{S}$$

for each of the following:

(i) $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

(ii) $\vec{F} = x\vec{i} + y\vec{j}$

We consider each problem individually.

(i) Consider the surface L which consists of S and the cap C on top of the cone oriented upward. Then the divergence Theorem implies

$$\int\int_L \vec{F}\cdot d\vec{S} = \int\int\int_I 3dV$$

where I is the interior of the cone. Therefore, we get

$$\int\int_L \vec{F}\cdot d\vec{S} = \int\int\int_I 3dV = 3\times\frac{1}{3}\pi 9\times 3 = 27\pi$$

i.e. 3 times the volume of the cone since the integral of the function 1 determines the volume of the region. This means

$$\int\int_S \vec{F}\cdot d\vec{S} = 27\pi - \int\int_C \vec{F}\cdot d\vec{S}$$

the latter integral which is much easier than the first. In particular, the cap is the part of the function $g(x,y) = 3$ over the circle D given by $x^2+y^2 \leqslant 9$, so we can use the flux formula for graphs of functions. Specifically, we have

$$\int\int_C \vec{F}\cdot d\vec{S} = \int\int_D (-x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} + 3)dA = \int\int_D 3dA = 27\pi$$

i.e. 3 times the area inside the circle D. In particular, we have

$$\int\int_S \vec{F}\cdot d\vec{S} = 0.$$

Note that we could have simply concluded this same result by observing that the vectors in the vector field $\vec{F}$ point along the surface S and do not flow through the surface!

(ii) We can follow a similar process as in the previous example. In particular, in this case, we shall get

$$\int\int_S \vec{F}\cdot d\vec{S} = 18\pi - \int\int_C \vec{F}\cdot d\vec{S}.$$

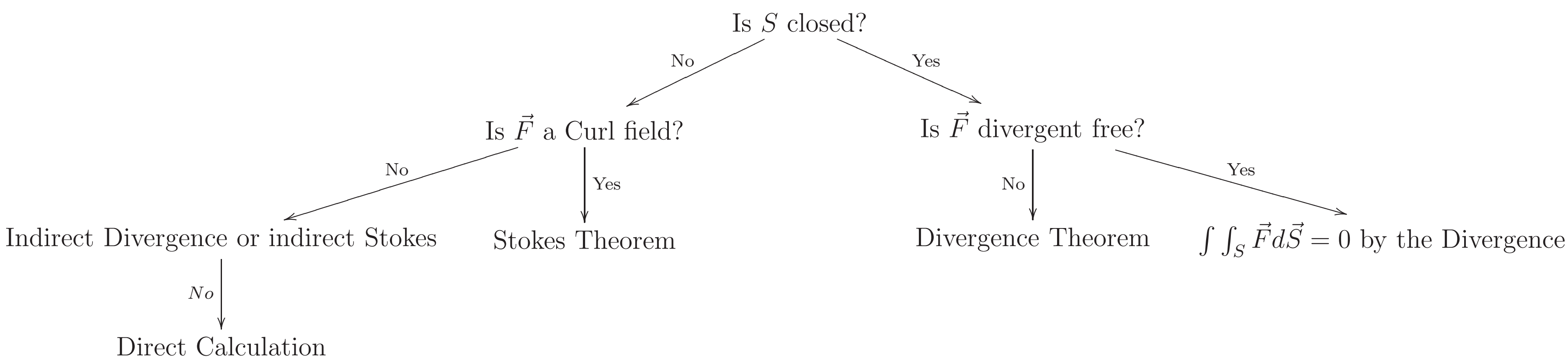
Observe however that

$$\int\int_C \vec{F}\cdot d\vec{S} = 0$$

since all the vectors in the vector field $\vec{F}$ point along the surface C (and not through the surface C. Hence

$$\int\int_S \vec{F} \cdot d\vec{S} = 18\pi.$$

We have learnt a lot of new material over the last few days, and as with line integrals, we have learnt many different ways to calculate them. Therefore, we finish, as we did with line integrals, with a decision tree of which questions to ask when trying to determine which method to use. As before, the general idea is to avoid using direct calculation as much as possible, and instead convert the problem using either Stokes theorem or the divergence theorem into a much simpler problem. Notice in the decision tree that only one path leads to direct calculation (at all costs, we want to avoid direct calculation since it is always a difficult process!).


Is S closed?
No
Yes
Is $\vec{F}$ a Curl field?
Is $\vec{F}$ divergent free?
No
Yes
No
Yes
Indirect Divergence or indirect Stokes
Stokes Theorem
Divergence Theorem
$\int\int_S \vec{F} d\vec{S} = 0$ by the Divergence
No
Direct Calculation