Section 1.2: Conditional Statements

In this section, we study one of the most important tools in logic - conditional statements. Loosely speaking, conditional statements are statements of the form "if p then q", and are in many respects the very foundation of most arguments.

1. The Conditional Statement

Before we give a formal definition of the conditional statement, we start with an example so we can understand when a conditional statement should be true. For the example, we need the following notation and terminolgy:

Notation 1.1. If p and q are statements, the **conditional** of q by p is "if p then q" denoted $p \to q$. We call p the hypothesis of the conditional and q the conclusion.

Example 1.2. Consider the conditional statement, "if I am healthy, I will come to class." To determine the truth value of this statement, we need to determine when this statement is false, so we consider the four different possibilities for the truth values of p and q. Let $p :=$"I am healthy" and $q :=$"I will come to class". We shall fill in the following table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- For case # 1, if I am healthy and I come to class, the conditional is clearly true.
- For case # 2, if I am healthy, but I have decided to stay home and not go to class, the conditional is false - the hypothesis is satisfied, but the conclusion is not satisfied, so the statement cannot possibly be true.
- For case # 3, if I am not healthy, but I have come to class anyway though all the people sitting around may not be happy about it, the conditional statement has not been violated since the hypothesis does not hold i.e. the conditional statement is meaningless since the hypothesis is not true. Therefore, the conditional must be true.
- Likewise, for case # 4, if I am not healthy, and I did not come to class, the conditional statement has not been violated since

> the hypothesis does not hold. Therefore, the conditional is true.

This example implies that a conditional statement is false only when the hypothesis is true and the conclusion is false. Though it is clear that a conditional statement is false only when the hypothesis is true and the conclusion is false, it is not clear why when the hypothesis is false, the conditional statement is always true. To try to explain why this is this case, we consider another example.

Example 1.3. Consider the mathematical statement "if n is a perfect square, then n is not prime." Clearly this is a true statement for any n, so it will be true when we substitute values in for n. Now substitute 3 for n:

"if 3 is a perfect square, then 3 is not prime."

As remarked above, this conditional statement is still true yet its hypothesis and conclusion are both false. Similarly, if we substitute 6 into this statement, it becomes

"if 6 is a perfect square, then 6 is not prime."

This conditional statement is true yet its hypothesis is false and its conclusion is true.

We can now write down a formal definition for the conditional statement.

Definition 1.4. If p and q are statements, the **conditional** of q by p is "if p then q" or "p implies q" denoted $p \to q$. A conditional statement is false only when the hypothesis is false and the conclusion is true. The truth table for the conditional statement is as follows:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 1.5. A conditional statement that is true by the fact that its hypothesis is false is often called vacuously true or true by default.

Remark 1.6. (Order of Operations) In expressions that include $\to$ as well other other logical operators, the order of operations is that $\to$ is always performed last.

2. Basic Examples using Conditional Statements

We consider some examples of how to work with conditional statements when trying to show logical equivalence or constructing basic truth tables.

Example 2.1. Show that the the statement $p\wedge \sim q \rightarrow p$ is a tautology.

p	q	$\sim q$	$p\wedge \sim q$	$p\wedge \sim q \rightarrow p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

This is a tautology since the truth values are always true.

Example 2.2. (Writing $\rightarrow$ as $\vee$) Show that $p \rightarrow q \equiv \sim p \vee q$.
We need to construct truth tables for these two statements. We have

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

and

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

They have the same truth values and thus they must be logically equivalent statements i.e. a statement of the form "if p then q" is equivalent to one of the form "either not p or q".

Example 2.3. Determine if the following statements are logically equivalent:

(i) "If you paid full price, you didn't buy it at Powells books"
(ii) "You didn't buy it at Powells books or you paid full price"

Let $p :=$"you paid full price" and $q :=$"you bought it at Powells books". The first statement is $p \rightarrow\sim q$ and the second statement is $\sim q \vee p$. We know $p \rightarrow\sim q$ is logically equivalent to $\sim p\vee \sim q$, and the second statement is equivalent to $p\vee \sim q$ (using the commutative law). Clearly these two statements are not equivalent and thus the statements above are not equivalent (even though they may initially seem so).

3. Logical Statements Related to Conditional Statements

There are many statements closely related to the conditional statement which are built up from conditional statements. We shall consider some of them in detail.

3.1. **The Negation of a Conditional Statement.** Recall that the conditional statement $p \to q$ is false only when p is true and q is false. Thus we get the following:

Result 3.1. The negation of "if p then q" is "p and not q". In symbolic notation,

$$\sim (p \to q) \equiv p \wedge \sim q$$

Example 3.2. The negation of the statement "If the river is narrow, then we can cross it quickly" is "The river is narrow and we cannot cross it quickly".

Warning. The negation of a conditional **not** a conditional - this is sometimes a misconception which can lead to bad logical inferences.

3.2. **The Contrapositive of a Conditional Statement.** One very important tool in mathematics and logic is the use of the contrapositive to prove arguments. The contrapositive is defined as follows.

Definition 3.3. The contrapositive of the conditional "if p then q" is the conditional "if not q then not p". In symbols, the contrapositive of $p \to q$ is $\sim q \to \sim p$.

The importance of the contrapositive of a conditional is that it is logically equivalent to the conditional. This means that a conditional statement holds only when its contrapositive holds. Though this may not currently seem helpful, we shall this is a very powerful tool in math. Before we do this, we shall prove a conditional is equivalent to its contrapositive.

Result 3.4. The conditional statement $p \to q$ is equivalent to its contrapositive $\sim q \to \sim p$.

Proof. We shall use logical equivalences:

$$p \to q \equiv \sim p \vee q \equiv q \vee \sim p \equiv \sim (\sim q) \vee \sim p \equiv \sim q \to \sim p$$

□

Example 3.5. Consider the statement "if xy is not even, then either x is not odd or y is not even". Determining the truth value of this statement seems fairly difficult, and on first glance, it looks like it may be false. We shall examine its truth value using the contrapositive. First, let
$p :=$ "xy is even",
$q :=$ "x is odd",
$r :=$ "y is even".
Then symbolically, the logical statement above is

$$\sim p \to \sim q \vee \sim r.$$

The contrapositive of this statement is

$$\sim (\sim q \vee \sim r) \rightarrow p$$

which is equivalent to

$$\sim (\sim (q \wedge r)) \rightarrow p$$

which in turn is equivalent to

$$q \wedge r \rightarrow p.$$

Translating back to language, this statement says "if x is odd and y is even, then xy is even" which is clearly true, and hence the original statement must be true.

3.3. **The Converse and Inverse of a Conditional Statement.** There are two other statements related to a conditional statement which on first glance one may think are equivalent to the conditional but are in fact **not**.

Definition 3.6. Suppose "if p then q" is a conditional statement. Then we define the following related statements:

(i) The converse of "if p then q" is "if q then p". In symbolic notation, the converse of $p \rightarrow q$ is $q \rightarrow p$.
(ii) The inverse of "if p then q" is "if not p then not q". In symbolic notation, the inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

The following result summarizes how these statements are related through logical equivalence.

Result 3.7. (i) A conditional statement is **not** equivalent to its converse.
(ii) A conditional statement is **not** equivalent to its inverse.
(iii) The inverse and converse of a conditional statement are equivalent to eachother.

Example 3.8. Consider the statement "If I am tall, then I can dunk a basketball". The converse to this statement is "If I can dunk a basketball, then I am tall", and the inverse is "If I am not tall, then I cannot dunk a basketball. Notice that these last two statements are logically equivalent.

3.4. **The "only if" Statement.** A statement closely related to a conditional statement is an "only if" statement i.e. a statement of the form "p only if q". To say "p only if q" means "if not q then not p" which is the contrapositive of "if p then q". Summarizing, we have:

Definition 3.9. If p and q are statements, then "p only if q" means "if not q then not p" or equivalently, "if p then q".

Example 3.10. The statement "I will get an A in this class only if I work hard" is logically equivalent to "If I get an A in this class, then I have worked hard in this class".

3.5. **The Biconditional Statement.** A statement related to the conditional and the "only if" statement is the so-called biconditional statement defined as follows:

Definition 3.11. If p and q are statements, then the biconditional of p and q is "p if only if q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if they have opposite truth values. The truth table of a biconditional statement is as follows:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Sometimes we abbreviate "if and only if" to "iff".

3.6. **Necessary and Sufficient Conditions.** Other phrases related to conditional statements are "necessary" and "sufficient" conditions. We summarize:

Definition 3.12. If p and q are statements:

- "p is a sufficient condition for q" means "if p then q"
- "p is a necessary condition for q" means "if not p then not q", or equivalently, "if q then p.

We finish with an example.

Example 3.13. The statement "doing homework regularly in this class is a necessary condition for achieving a good grade" is logically equivalent to "if homework is not done regularly in this class, then a good grade will not be acheived" or equivalently, "if a good grade is achieved in this class, then homework was done regularly".

Homework

(i) From the book, pages 27-29: Questions: 3, 10, 15, 17, 19, 20a, 20e, 20f, 25, 26, 33, 37, 39, 44, 46, 47, 48