Section 2.2: Introduction to the Logic of Quantified Statements

In this section, we shall continue to examine some of the fundamentals of predicate calculus. Specifically, we shall look at the negations of quantified statements, consider conditional statements in more detail (and the variants of conditional statements), and look at other related statements.

1. Negations of Quantified Statements

When we defined quantified statements, we already considered their negations, so we shall briefly summarize them and then look at some examples.

Theorem 1.1. *The negation of a statement*

$$\forall x \in D, P(x)$$

is equivalent to

$$\exists x \in D \text{ such that } \sim P(x).$$

Theorem 1.2. *The negation of a statement*

$$\exists x \in D, P(x)$$

is equivalent to

$$\forall x \in D, \sim P(x).$$

In summary, **the negation of an existential statement is a universal statement** and **the negation of a universal statement is an existential statement**. We illustrate with some examples.

Example 1.3. Negate the following statements informally:

(i) "All cars are red"

The negation would be "There exists a car which is not red" or, "there are cars which are not red"

(ii) "Some birds cannot fly"

The negation would be "all birds can fly".

Example 1.4. Write the following statements formally and then negate them:

(i) "Some math classes are not fun"

Let D be the set of all math classes and $P :=$ "is fun". Then formally, this statement is:

$$\exists x \in D, \sim P(x).$$

The formal negation of this statement is

$$\forall x \in D, \sim\sim P(x)$$

which is equivalent to

$$\forall x \in D, P(x).$$

(*ii*) "All math classes are fun"

Let D be the set of all math classes and $P :=$ "is fun". Then formally, this statement is:

$$\forall x \in D, P(x).$$

The formal negation of this statement is

$$\exists x \in D, \sim P(x).$$

We can use the rules for negating existential and universal statements together with De Morgans laws and our knowledge of the negations of the other basic propositional connectives to negate much more difficult statements. We illustrate with an example.

Example 1.5. Negate the following statements:

(*i*) $\forall x, P(x) \wedge Q(x)$

We have

$$\sim (\forall x, P(x) \wedge Q(x)) \equiv \exists x, \sim (P(x) \wedge Q(x)) \equiv \exists x, \sim P(x) \vee \sim Q(x)$$

(*ii*) $\forall x, P(x) \wedge Q(x) \to R(x)$

We have

$$\begin{aligned} \sim (\forall x, P(x) \wedge Q(x) \to R(x)) &\equiv \exists x, \sim (P(x) \wedge Q(x) \to R(x)) \\ &\equiv \exists x, (P(x) \wedge Q(x)) \wedge \sim R(x) \end{aligned}$$

since we are negating a conditional statement.

2. Universal Conditional Statements

As with the regular conditional statement, there are a number of differences between universal conditional statements and regular universal statements, so we consider them separately here.

2.1. **Negating a Universal Conditional Statement.** In the last example we considered, we negated a universal conditional statement. Since it is a fairly important technique in math and logic, we formalize our answer.

Theorem 2.1. *The negation of the universal conditional statement* $\forall x, P(x) \to Q(x)$ *is the existential statement* $\exists x, P(x) \wedge \sim Q(x)$*.*

We illustrate with an example.

Example 2.2. Negate the statement "all red cars are fast"

The negation of this statement will be "there exists a car which is red and which is not fast".

2.2. **Related Conditional Statements.** Just as with the regular conditional statement, there are a few closely related statements to the universal conditional statement.

Definition 2.3. Consider the universal conditional statement

$$\forall x \in D, P(x) \to Q(x)$$

(i) The contrapositive of this statement is

$$\forall x \in D, \sim Q(x) \to \sim P(x)$$

(ii) The converse of this statement is

$$\forall x \in D, Q(x) \to P(x)$$

(iii) The inverse of this statement is

$$\forall x \in D, \sim P(x) \to \sim Q(x)$$

In general, the contrapositive of a universal conditional statement is equivalent to a statement and the inverse and converse of a universal conditional statement are equivalent. However, as statement and its converse are **not** logically equivalent.

We illustrate with some examples.

Example 2.4. Consider the statement "if an integer x does not equal 0 or 1, then $x^2 > x$". Write the statement as a universal conditional statement and state the contrapositive, converse and inverse.

Universal conditional statement: "$\forall$ integers x, if $x \neq 0$ and $x \neq 1$ then $x^2 > x$".

Contrapositive: "$\forall$ integers x, if $x^2 \leqslant x$ then either $x = 0$ or $x = 1$".

Converse: "$\forall$ integers x, if $x^2 > x$ then $x \neq 0$ and $x \neq 1$".

Inverse: "$\forall$ integers x, if either $x = 0$ or $x = 1$ then $x^2 \leqslant x$".

Notice that the converse and inverse are certainly not logically equivalent to the original statement since the original statement is false (take $x = 1/2$ as a counter example) and the converse is true.

2.3. **Vacuous Truth Of Universal Statements.** Just as with the regular conditional statement, a universal conditional statement can be vacuously true or true by default. Specifically, we have the following:

Result 2.5. Consider the universal conditional statement

$$\forall x \in D, P(x) \to Q(x).$$

If $P(x)$ is false for all $x \in D$, then the universal conditional statement is true. In such circumstances, we say it is true by default or vacuously true.

This seems like a silly statement to make, but we need predicate logic to be consistent with propositional logic. Moreover, when we consider examples, it is more clear why it seems reasonable.

Example 2.6. Consider the statement "$\forall$ integers x, if x is odd and even, then x is made of cheese." Clearly this is absurd, there is no way that an integer can be made of cheese, but if we look at a little closer, we shall see this is not the case.
To show this statement is false, we would need to show its negation is true i.e "$\exists$ an integer x, such that x is odd and even, and x is not made of cheese." However, notice that there are no even and odd integers i.e. if "$P :=$ is odd and even", then $P(x)$ is false for every integer x. Thus this existential expression can never be true (since there are no even and odd integers). It follows that the original statement must be true (since its negation was false)

3. Necessary and Sufficient Conditions, Only if

The definitions we introduced for "necessary", "sufficient" and "only if" can also be extended to universal statements.

Definition 3.1.
- (*i*) The statement "$\forall x, r(x)$ is a sufficient condition for $s(x)$" means "$\forall x, r(x) \to s(x)$".
- (*ii*) The statement "$\forall x, r(x)$ is a necessary condition for $s(x)$" means "$\forall x, \sim r(x) \to \sim s(x)$" or equivalently "$\forall x, s(x) \to r(x)$".
- (*iii*) The statement "$\forall x, r(x)$ only if $s(x)$" means "$\forall x, \sim s(x) \to \sim r(x)$" or equivalently "$\forall x, r(x) \to s(x)$".

We illustrate with some examples.

Example 3.2. Rewrite the following statements in formal logic (the domain is the set of all complex numbers denoted $\mathbb{C}$) and in informal language.

- (*i*) $x^2 < 0$ only if x is an imaginary number

  Let $P(x) := x^2 < 0$ and $Q :=$"is an imaginary number". Then this formally translates to "$\forall x \in \mathbb{C}, \sim Q(x) \to \sim P(x)$" or "$\forall$ complex numbers, if x is not imaginary, then $x^2 \geqslant 0$"
- (*ii*) $x^2 < 0$ is a sufficient condition for x to be a imaginary number

  Let $P(x) := x^2 < 0$ and $Q :=$"is an imaginary number". Then this formally translates to "$\forall x \in \mathbb{C}, P(x) \to Q(x)$" or "$\forall$ complex numbers, if $x^2 < 0$, then x is imaginary"
- (*iii*) $x^2 < 0$ is a necessary condition for x to be a imaginary number

  Let $P(x) := x^2 < 0$ and $Q :=$"is an imaginary number". Then this formally translates to "$\forall x \in \mathbb{C}, \sim P(x) \to \sim Q(x)$" or "$\forall$ complex numbers, if $x^2 \geqslant 0$, then x is not imaginary

Homework

(i) From the book, pages 95-97: Questions: 2, 3, 4, 8, 9, 13, 15, 19, 25, 28, 32, 37, 38, 42, 43