Surface Area

In this section, we shall use Riemann sums and what we know about arc length to help develop a formula for the surface area of some solid.

1. Determining Surface Area Using Riemann Sums

The set up will be as follows:

(i) Suppose \( f(x) > 0 \) on an interval \([a, b]\) and \( f(x) \) is continuous. Let \( S \) be the solid obtained by rotating the graph of \( f(x) \) about the \( x \)-axis.

(ii) We break the interval \([a, b]\) up into smaller intervals of lengths \( \Delta x \). Let \([x_i, x_{i+1}]\) be the \( i \)-th interval. We shall approximate the surface area of the strip obtained by rotating the region of the graph on this interval around the \( x \)-axis.

(iii) Observe that this strip is not quite cylindrical shaped, so we cannot use the formula for the surface area of a cylinder. In fact when we cut it and roll it out, we get a slightly curved strip of paper as illustrated:

If we straighten out this piece of paper, the length of this strip can be approximated by \( 2\pi f(x_i) \) (since \( f(x_i) \) is the approximate radius of the circle). The width of the strip will not be \( \Delta x \) but rather the length of the curve from \( x_i \) to \( x_{i+1} \), which can be approximated by \( \sqrt{1 + (f'(x_i))^2} \Delta x \) (as we saw last section).

(iv) It follows that the surface area of this small strip can be approximated by \( 2\pi f(x_i) \sqrt{1 + (f'(x_i))^2} \Delta x \). Summing up over all strips and letting \( \Delta x \to 0 \), the approximation becomes more exact, so we get the following result.

Result 1.1. If \( f(x) \) is a continuous function on \([a, b]\) and \( f(x) \geq 0 \), then the surface area of the solid obtained by rotating the graph of
$f(x)$ about the $x$-axis between $[a, b]$ is

$$\int_a^b 2\pi f(x) \sqrt{1 + x^2} \, dx.$$ 

We look at a couple of examples to illustrate our result.

**Example 1.2.** Derive the surface area formula for a sphere of radius $R$.

We can construct the sphere of radius $R$ by rotating the half circle $y = \sqrt{R^2 - x^2}$ about the $x$-axis. Observe that $dy/dx = -2x/\sqrt{R^2 - x^2}$, so

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + 4x^2/(R^2 - x^2)} = \sqrt{R^2/(R^2 - x^2)} = R/\sqrt{R^2 - x^2}.$$ 

Using the formula, we get

$$\text{Surface Area} = \int_{-R}^R 2\pi \sqrt{R^2 - x^2} \frac{R}{\sqrt{R^2 - x^2}} \, dx = \int_{-R}^R 2\pi R \, dx = 2\pi R[R - (-R)] = 4\pi R^2$$

As with all other things we have considered, we can also consider functions of $y$ instead of $x$ and the formula still holds.

**Example 1.3.** Calculate the surface area of the surface obtained by rotating $y = x^{1/3}$, $1 \leq y \leq 2$ about the $y$-axis.

Observe that $x = y^3$, so $dx/dy = 3x^2$ and

$$\sqrt{1 + (dx/dy)^2} = \sqrt{1 + (3y^2)^2} = \sqrt{1 + 9y^4}.$$ 

Applying the formula, we get

$$\int_1^2 2\pi y^3 \sqrt{1 + 9y^4} \, dy = \frac{\pi}{27} \left[ 1 + 9x^4 \right]^{3/2} \bigg|_1^2 = \frac{\pi}{27} \left[ 145^{3/2} - 10^{3/2} \right].$$

2. **Applications - Gabriels Horn**

In mathematics, the concept of infinite is very strange. We look at an example now of a surface with finite volume but infinite surface area. This contradicts all logic because it means you can "paint" an infinite amount of surface with a finite amount of paint (just fill up Gabriels horn and let it seep through):

**Example 2.1.** Gabriels horn is the solid obtained by rotating the equation $y = 1/x$ for $x \geq 1$ about the $x$-axis. Show that the volume of Gabriels horn is finite, but the surface area is infinite. Hence find out how may liters of paint are needed to paint a wall with infinite surface area.

To calculate volume, we use washers:

$$\int_1^\infty \frac{1}{x^2} \, dx = \lim_{t \to \infty} \left[ -\frac{\pi}{x} \right]^t_1 = \lim_{t \to \infty} \left[ \pi - \frac{\pi}{t} \right] = \pi.$$
To calculate volume, we use the formula observing that $f'(x) = -1/x^2$ and $\sqrt{1 + (f'(x))^2} = \sqrt{1 + 1/x^4} = \sqrt{x^4 + 1/x^2}$:

$$\int_1^\infty 2\pi \frac{1}{x} \frac{\sqrt{x^4 + 1}}{x^2} \, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3} \, dx.$$ 

Since this is a definite integral which is always positive, we can bound it above. Observe that

$$\frac{\sqrt{x^4 + 1}}{x^3} \geq \frac{x^2}{x^3} = \frac{1}{x},$$

and thus

$$2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3} \, dx \geq 2\pi \int_1^\infty \frac{1}{x} \, dx$$

which diverges (i.e., it goes to $\infty$). Thus the surface area is infinite, but we would only need $\pi$ liters of paint to paint a wall with infinite surface area.