Section 13.5
Equations of Lines and Planes

“Generalizing Linear Equations”
One of the main aspects of single variable calculus was approximating graphs of functions by lines - specifically, tangent lines. In multivariable calculus, we generalize this idea to the graphs of multivariable functions. In order to do this, first we need to be able to write down the equations for lines in 3-space, and then we shall examine how to write down equations for planes.

1. EQUATIONS OF LINES
In 2-space, the equation for a line is simply a linear equation involving two variables. In 3-space however, this is no longer the case as illustrated by the following example.

Example 1.1. Suppose $y = x$.

(i) Draw the graph of this equation in 2-space.

(ii) Draw the graph of this equation in 3-space.

Note that this is a plane, not a line!
Instead of just using algebraic equations as we did in 2-space, we need a new way to represent lines in 3-space. There are three different ways we shall use to describe a line, each of which have advantages.

1.1. Vector Equations for Lines. Recall that a line is completely determined by two things - the direction it is pointing and some point on the line. Once these two things are given, the line is completely determined. Since vectors describe a direction, we can use a vector and a point in space to determine a line. We do it as follows:

- Suppose \( L \) is a line passing through the two points \( P(a, b, c) \) and \( Q(x, y, z) \).
- Let \( \vec{v} \) be the displacement vector from \( P \) to \( Q \).
- Define the “vector equation”
  \[
  \vec{r}(t) = \vec{P} + \vec{v}t
  \]
  where \( \vec{P} \) denotes the position vector with head at the point \( (a, b, c) \).
- We claim that \( \vec{r}(t) \) describes the line \( L \) completely as \( t \) varies over the real numbers. To see this, observe that when \( \vec{r}(0) = \vec{P} \), so the equation \( \vec{r}(t) \) passes through the point \( P \). Then as \( t \) increases or decreases, the equation \( \vec{r}(t) \) passes through points which lie in the direction determined by \( \vec{v} \) from the point \( P \), and these are exactly the points on \( L \).

We summarize:

**Result 1.2.** Suppose \( L \) is a line passing through \( P \) and \( Q \). If \( \vec{v} \) is the displacement vector from \( P \) to \( Q \), then a vector equation for the line \( L \) is

\[
\vec{r}(t) = \vec{P} + \vec{v}t
\]

(notice that this equation is linear in \( t \)).

**Example 1.3.** Write down a vector equation for the line with direction vector \( 2\vec{i} + 2\vec{j} - \vec{k} \) passing through \((2, 2, 2)\). Use it to write down two other points it passes through.

We have

\[
\vec{r}(t) = (2 + 2t)\vec{i} + (2 + 2t)\vec{j} + (2 - t)\vec{k}
\]

To find points on \( L \), we simply plug in values for \( t \). Putting in \( t = 1 \) and \( t = 2 \), we get the point \((3, 3, 1)\) and \((4, 4, 0)\).

1.2. Parametric Equations for a Line. Another way to represent a line in 3-space is through the use of parametric equations for the coordinates \( x \), \( y \) and \( z \). Specifically, if \( \vec{r}(t) = \vec{P} + \vec{v}t \) is a vector equation for our line where \( \vec{P} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} \) and \( \vec{v} = a\vec{i} + b\vec{j} + c\vec{k} \), we can combine these vectors to get

\[
\vec{r}(t) = (at + x_0)\vec{i} + (bt + y_0)\vec{j} + (ct + z_0)\vec{k}
\]

Since the component of \( \vec{i} \) is the \( x \)-coordinate on the line, the component
of $\vec{j}$ is the $y$-coordinate on the line, and the component of $\vec{k}$ is the $z$-coordinate on the line, we can read off equations for $x$, $y$ and $x$ in terms of $t$. Specifically, we get:

**Result 1.4.** Suppose $L$ is a line passing through $P(x_0, y_0, z_0)$ with direction vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$. Then the three scalar equations

\[
x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct
\]
determine the line $L$ where $t$ is a parameter varying over all real numbers. They are called **parametric equations** for the line $L$.

**Example 1.5.** Find a vector equation and parametric equations for the line through the points $P(1, 2, 3)$ and $Q(3, 2, 1)$. Use it to determine whether this line passes through the origin.

Displacement vector is $\vec{v} = -2\vec{i} + 2\vec{k}$, and we can take $\vec{P} = \vec{i} + 2\vec{j} + 3\vec{k}$ as our point on the line. Then we have

\[
\vec{r}(t) = (1 - 2t)\vec{i} + 2\vec{j} + (3 + 2t)\vec{k}
\]
as the vector equation and

\[
x = 1 - 2t \quad y = 2 \quad z = 3 + 2t
\]
as parametric equations. Clearly the line does not pass through the origin because the $y$ coordinate can never be $0$.

If the vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ is the direction vector for a line, we sometimes refer to $a$, $b$, and $c$ as **direction numbers**. Note that if $a$, $b$, and $c$ are direction numbers for a line and $k$ is a constant, then $ka$, $kb$, and $kc$ are also direction numbers for the line (WHY?).

### 1.3. Symmetric Equations for Lines

One last way to represent a line is through a system of equalities. Specifically, provided $a$, $b$, and $c$ are not $0$, we can solve each of the parametric equations for a line $L$ for $t$, and then all resulting equations must be equal. Specifically, we have:

**Result 1.6.** The symmetric equations for a line $L$ with parametric equations

\[
x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct
\]
are

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\]

**Example 1.7.** Find symmetric equations for the line through the points $P(1, 2, 3)$ and $Q(3, 0, 1)$ and use it to determine where this line passes through the $xy$-plane.

The parametric equations for this line are

\[
x = 1 - 2t \quad y = 2 + 2t \quad z = 3 + 2t
\]
so the symmetric equations will be
\[\frac{-x - 1}{2} = \frac{y - 2}{2} = \frac{z - 3}{2}.\]
It travels through the xy-plane when \(z = 0\), so
\[\frac{-x - 1}{2} = -\frac{3}{2}\]
and
\[\frac{y - 2}{2} = -\frac{3}{2}\]
giving, \(x = 4\) and \(y = -1\). or \((4, -1, 0)\).

1.4. Line Segments. Sometimes we may only be interested in a small portion of a line - a line segment. If we want to write down the equation for the line segment between two points \(P\) and \(Q\) starting at \(P\), we note that the direction vector between \(P\) and \(Q\) can be written as \(\vec{Q} - \vec{P}\), so then the vector equation can be written as
\[\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P})\]
or
\[\vec{r}(t) = (1 - t)\vec{P} + t\vec{Q}.\]
Observe that when \(t = 0\), this is at \(P\) and when \(t = 1\), it is at \(Q\). Thus we have:

**Result 1.8.** The line segment between two points \(P\) and \(Q\) starting at \(P\) is given by the vector equation
\[\vec{r}(t) = (1 - t)\vec{P} + t\vec{Q}\]
where \(0 \leq t \leq 1\).

We finish with another slightly more general example.

**Example 1.9.** Show that the lines \(L_1\) and \(L_2\) with parametric equations
\[x = 1 - 2t \quad y = 2 + 2t \quad z = 3 + 2t\]
and
\[\vec{r}(t) = (2 + 2t)i + (2 + 2t)j + (2 - t)k\]
respectively never intersect.

First we note that to say they intersect means they pass through the same point, though not necessarily at the same time. Thus we are asking if there is a value \(s\) and \(t\) such that \(1 - 2t = 2 + 2s\), \(2 + 2t = 2 + 2s\), and \(3 + 2t = 2 - s\). Solving these equations, we get \(t = s\) from the second equation, and then \(1 - 2s = 2 + 2s\) and \(3 + 2s = 2 - s\) from the remaining two. Since these remaining equations have no solutions, there can be no value of \(s\) and \(t\) satisfying these conditions and thus no point of intersection.
Example 1.10. Do non-parallel lines have to intersect?

In 2-space, we know that two non-parallel lines have to intersect. However, in 3-space, this is not the case. Take for example $L_1$ with vector equation $\vec{r}_1(t) = t\vec{i} + \vec{k}$ and $L_2$ with vector equation $\vec{r}_2(t) = (t)\vec{j}$. As before, they will intersect if they pass through the same point. However, the $\vec{k}$ component of $\vec{r}_2(t)$ and the $\vec{k}$ component of $\vec{r}_1(t)$ will always be 1. In particular, they can never have the same $\vec{k}$ component and hence will never cross through the same $z$-value, so can never cross through the same point.

2. Equations of Planes

The 2-dimensional equivalent of a line is a plane. As with the line, we shall now determine different ways to write down the equation for a plane.

2.1. Vector Equation for a Plane. To determine a vector equation for a plane, we cannot simply take a vector and a point in the plane as we did with the line because there are many vectors in a plane pointing in lots of different directions. However, instead of taking a vector in the plane, we can take a vector perpendicular (called a normal vector) to the plane - in 3-space, there is only one direction (up to scalar multiplication) in which a vector perpendicular to a plane can point (either up or down from the plane). Therefore we can construct the vector equation for a plane as follows:

- Let $P(x_0, y_0, z_0)$ be a point in the plane $K$ and let $n = ai + bj + ck$ be a normal vector to $P$.
- If $Q(x, y, z)$ is any other point in the plane, let $\vec{PQ} = \vec{P} - \vec{Q}$ be the displacement vector from $P$ to $Q$.
- Note that $\vec{P} - \vec{Q}$ lies entirely in the plane, so will be perpendicular to $\vec{n}$. In particular, we shall have $\vec{n} \cdot (\vec{P} - \vec{Q}) = 0$ or $\vec{n} \cdot \vec{P} = \vec{n} \cdot \vec{Q}$.
- Thus any arbitrary point $Q(x, y, z)$ in $K$ must satisfy the equation $\vec{n} \cdot \vec{P} = \vec{n} \cdot \vec{Q}$.

Thus we have proved the following:

Result 2.1. If $K$ is a plane with normal vector $\vec{n}$ and $P$ is a point in $K$ then any arbitrary point $Q$ in $K$ must satisfy the equation

$$\vec{n} \cdot \vec{P} = \vec{n} \cdot \vec{Q}.$$  

We call this equation the vector equation for the plane $K$.

Example 2.2. Find a vector equation for the plane which contains the points $P(1, 2, 3)$, $Q(2, 4, 1)$ and $R(-1, 2, -3)$. 

In order to find a vector equation, we need a point and a normal vector. We already have a point (we can choose any of the three). In order to find a normal vector, we can use the cross-product on the displacement vectors on the points in the plane.

\[
P\vec{Q} \times P\vec{R} = (\vec{i} + 2\vec{j} - 2\vec{k}) \times (-2\vec{i} - 6\vec{k}) = -12\vec{i} + 10\vec{j} + 4\vec{k}
\]

Taking the point \( P(1, 2, 3) \), the vector equation for the plane will be

\[
(12\vec{i} + 10\vec{j} + 4\vec{k}) \cdot (\vec{i} + 2\vec{j} + 3\vec{k}) = (12\vec{i} + 10\vec{j} + 4\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k})
\]

where \( x\vec{i} + y\vec{j} + z\vec{k} \) is any point in the plane.

2.2. Scalar Equation for a Plane. The vector equation for a plane is rarely used - in fact scalar equations for planes are fairly easy to determine and easy to write down (unlike scalar equations for lines). We can use the vector equation to write down the scalar equation.

**Result 2.3.** If \( \vec{n} = a\vec{i} + b\vec{j} + c\vec{k} \) is a normal vector to a plane \( K \) and \( P(x_0, y_0, z_0) \) is a point in \( K \), then an arbitrary point \( Q(x, y, z) \) in \( K \) satisfies the equation

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]

called the **scalar equation** for the plane \( K \). This equation can also be written as

\[
ax + by + cz + d = 0
\]

by combining constant terms and this equation is called the **linear equation** for \( K \).

It is easy to determine this equation. We know the each point \( Q \) on the plane satisfies

\[
\vec{n} \cdot \vec{P} = \vec{n} \cdot \vec{Q}.
\]

Calculating, we get

\[
ax_0 + by_0 + cz_0 = (a\vec{i} + b\vec{j} + c\vec{k}) \cdot (x_0\vec{i} + y_0\vec{j} + z_0\vec{k})
\]

and

\[
(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) = ax + by + cz
\]

so

\[
ax_0 + by_0 + cz_0 = ax + by + cz
\]

or

\[
a(x - x_0) + b(y - y_0) + c(x - z_0) = 0.
\]

If we let \( d = -ax_0 - by_0 - cz_0 \), then we get the linear equation \( ax + by + cz + d = 0 \). We illustrate with some examples.
**Example 2.4.** Find the equation of the plane through the point \((-2, 3, 8)\) and parallel to the plane \(2x + y + z - 1 = 0\).

We need to find a normal vector. Since the plane we are looking for is parallel to \(2x + y + z - 1 = 0\), it has the same normal vector - namely \(2\vec{i} + \vec{j} + \vec{k}\) (WHY?). Thus, the equation for the plane will be \(2(x + 2) + (y - 3) + (z - 8) = 0\) or \(2x + y + z - 7 = 0\).

3. Examples

We consider some examples combining the results we have developed.

**Example 3.1.** Find the equation of the line which is the intersection of the planes \(2x + y + z - 1 = 0\) and \(-2x + 2y - z = 0\).

In order to determine the equation for a line, we need a vector which points in the direction of the line and a point on the line. First note that a point on the line will simply be a point common to both planes. To find a point of intersection, we note that there will be infinitely many, so to make things a little easier, we shall find the point of intersection in the \(xy\)-plane (since they are not parallel to this plane, they must both intersect this plane and hence so must the intersection). This means \(z = 0\), so we get \(2x + y = 1\) and \(-2x + 2y = 0\). Solving, we have \(x = 1/3\) and \(y = 1/3\) (so the point of intersection is \((1/3, 1/3, 0)\).

Now we need to find a vector pointing in the same direction as the line. Note that a normal vector for the first plane will be \(\vec{n}_1 = 2\vec{i} + \vec{j} + \vec{k}\) and a normal vector for the second plane will be \(\vec{n}_2 = -2\vec{i} + 2\vec{j} - \vec{k}\). To find a vector which points in the direction of the intersection, we observe that it must lie in both planes, so will be perpendicular to both the normal vectors the planes. Such a vector can be found using the cross product:

\[
\vec{n}_1 \times \vec{n}_2 = (2\vec{i} + \vec{j} + \vec{k}) \times (-2\vec{i} + 2\vec{j} - \vec{k}) = -2\vec{i} + 4\vec{k}.
\]

Putting this information together, we get the line

\[
\vec{r}(t) = (1/3 - 2t)\vec{i} + 1/3\vec{j} + 4t\vec{k}
\]

**Example 3.2.** Find where the line

\[
\vec{r}(t) = (2 - 3t)\vec{i} + 8\vec{j} + 4t\vec{k}
\]

intersects the plane \(2x + y + z - 1 = 0\).

In order to do this, we substitute the values in for \(x, y\) and \(z\):

\[
2x + y + z - 1 = 2(2 - 3t) + 8 + 4t - 1 = 4 - 6t + 8 + 4t = -2t + 12 = 0,
\]

so \(t = 6\). This gives the point \((-16, 8, 24)\) as the point of intersection.
4. Application: Distance from a Point to a Plane

One important application is finding the distance between a point and a plane or between two parallel planes. We note that the shortest path between a point and a plane will be along the normal vector $\vec{n}$ of a plane. Thus if $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ is the normal vector to a plane and $(x_0, y_0, z_0)$ is an arbitrary point in a plane, the shortest path will be along the line through $(x_0, y_0, z_0)$ in the direction of $\vec{n}$. This can be calculated fairly easily, and in general, we get the following:

**Result 4.1.** The distance $D$ from a point $(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$ is given by the formula

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

**Example 4.2.** Find the distance between the parallel planes $x + y + 2z - 1 = 0$ and $-5x - 5y - 10z = 0$.

Observe that the point $(-1, 0, 0)$ is in the first plane. Since the shortest distance will the same between all points, we just need to find the distance between this point and the plane. Applying the formula, we have:

$$D = \frac{|-5(-1)|}{\sqrt{25 + 25 + 100}} = \frac{5}{\sqrt{150}}$$