Section 7.2: Trigonometric Integrals

1. Basic Trigonometric Integrals and Identities

In this section, we approach the problem of evaluating trigonometric integrals (integrals involving powers and sums of the basic trigonometric functions). Before we do this, we recall a few facts that will be extremely useful.

(i) The basic trigonometric integrals we need:

(a) \[ \int \sin (x)dx = - \cos (x) + C \]

(b) \[ \int \cos (x)dx = \sin (x) + C \]

(c) \[ \int \tan (x)dx = \ln | \sec (x) | + C \]

(d) \[ \int \sec (x)dx = \ln | \sec (x) + \tan (x) | + C \]

(ii) The basic trig identities we need.

(a) \[ \sin^2 (x) + \cos^2 (x) = 1 \]

(b) \[ \sec^2 (x) = 1 + \tan^2 (x) \]

(c) \[ \cos^2 (x) = \frac{1}{2}(1 + \cos (2x)) \]

(d) \[ \sin^2 (x) = \frac{1}{2}(1 - \cos (2x)) \]

(e) \[ \sin (x) \cos (x) = \frac{1}{2} \sin (2x) \]

(f) \[ \sin (A) \cos (B) = \frac{1}{2}[\sin (A - B) + \sin (A + B)] \]

(g) \[ \sin (A) \sin (B) = \frac{1}{2}[\cos (A - B) - \cos (A + B)] \]
(h) \[ \cos (A) \cos (B) = \frac{1}{2} [\cos (A - B) + \cos (A + B)] \]

It is not imperative that you memorize all of these formulas since there are a lot, and many are very similar so it could cause confusion. On a test, you will be able to use a sheet which includes these (and a few other) formulas. What is important is learning the techniques involved for a trigonometric integral. There are many circumstances which are very clear cut, which we shall discuss in detail. However, there are even more cases which are not quite so clear cut and require some non-standard reasoning. The basic idea behind any trigonometric integral is the following:

(i) Try to simplify the integrand using trigonometric identities to obtain an integrand on which you can apply integration by substitution. This may require a number of steps or may fall into one of a few special cases.

(ii) Complete the integral using integration by substitution.

As you can probably imagine, a bulk of the work comes from the first step of simplification, and it is what we shall spend most time on. We shall proceed through cases.

2. Evaluating \( \int \sin^m (x) \cos^n (x) \, dx \)

This falls into different cases depending upon whether the powers of \( \cos (x) \) and \( \sin (x) \) are even or odd.

(i) When the power of \( \cos (x) \) is odd, we substitute \( \cos^2 (x) = 1 - \sin^2 (x) \) to eliminate all but a single power of \( \cos (x) \) and then substitute \( u = \sin (x) \).

Example 2.1.

\[
\int \cos^3 (x) \sin^4 (x) \, dx = \int \cos (x)(1 - \sin^2 (x))(\sin^2 (x)) \, dx
\]

\[
= \int \cos (x)(\sin^2 (x) - \sin^4 (x)) \, dx
\]

Letting \( u = \sin (x) \), we get \( du/dx = \cos (x) \) or \( dx = du/\cos (x) \), so

\[
= \int \cos (x)(\sin^2 (x) - \sin^4 (x)) \, dx = \int \cos (x)(u^2 - u^4) \frac{du}{\cos (x)}
\]

\[
= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 (x)}{3} - \frac{\sin^5 (x)}{5} + C
\]
(ii) When the power of \( \sin(x) \) is odd and the power of \( \cos(x) \) is even, we substitute \( \sin^2(x) = 1 - \cos^2(x) \) to eliminate all but a single power of \( \sin(x) \) and then substitute \( u = \cos(x) \). It works in exactly the same way as the previous case, so we shall not go over another example.

(iii) When both power of \( \cos(x) \) and the power of \( \sin(x) \) is even, we use half angle identities repeatedly to reduce the trigonometric equation into something we can integrate.

**Example 2.2.**

\[
\int \sin^2(x) \cos^2(x) \, dx = \int \frac{1}{4} (1 - \cos(2x))(1 + \cos(2x)) \, dx
\]

\[
= \frac{1}{4} \int 1 + \cos^2(2x) \, dx = \frac{1}{4} \int (1 + \frac{1}{2}(1 + \cos(4x))) \, dx
\]

\[
= \frac{1}{8} \int 3 + \cos(4x) \, dx = \frac{3x}{8} + \frac{\sin(4x)}{32} + C
\]

3. **Evaluating** \( \int \tan^n(x) \sec^n(x) \, dx \)

This falls into different cases depending upon whether the powers of \( \sec(x) \) and \( \tan(x) \) are even or odd. For the case which do not fall into one of the two categories below, there is no single clear cut way to evaluate the given integral.

(i) When the power of secant is even, save a factor of \( \sec^2(x) \) and use \( \sec^2(x) = 1 + \tan^2(x) \) to write all other factors in terms of \( \tan(x) \). Then substitute \( u = \tan(x) \).

**Example 3.1.**

\[
\int \sec^4(x) \tan^3(x) \, dx = \int \sec^2(x)(1 + \tan^2(x)) \tan^3(x) \, dx
\]

\[
= \int \sec^2(x)(\tan^3(x) + \tan^5(x)) \, dx
\]

Substituting \( u = \tan(x) \), we get \( dx = \frac{du}{\sec^4(x)} \), and

\[
\int \sec^2(x)(\tan^3(x) + \tan^5(x)) \, dx = \int (u^3 + u^5) \, du = \frac{u^4}{4} + \frac{u^6}{6} + C
\]

\[
= \frac{\tan^4(x)}{4} + \frac{\tan^6(x)}{6} + C
\]

(ii) If the power of tangent is odd, save a factor of \( \sec(x) \tan(x) \) and use \( \tan^2(x) = \sec^2(x) - 1 \) to write all other factors in terms of \( \sec(x) \). Then substitute \( u = \sec(x) \).
Example 3.2.

\[ \int \tan^3(x) dx = \int \tan(x)(\sec^2(x) - 1) dx \]

\[ = \int \sec(x) \tan(x) \sec(x) dx - \int \tan(x) dx \]

The second integral can be evaluated explicitly. The first integral requires the substitution \( u = \sec(x) \), in which case \( dx = \frac{du}{\sec(x) \tan(x)} \) (WHY?). Thus,

\[ \int \sec(x) \tan(x) \sec(x) dx - \int \tan(x) dx = \int u du - \ln |\sec(x)| \]

\[ = \frac{u^2}{2} - \ln |\sec(x)| + C = \frac{\sec^2(x)}{2} - \ln |\sec(x)| + C \]

4. Evaluating \( \int \cos(mx) \sin(nx) dx \) and other such integrals

For these types of integrals, we use the sum and difference formulas. Since they all follow the same idea, we illustrate with just one example.

Example 4.1.

\[ \int \sin(3x) \cos(4x) dx = \int \frac{1}{2} (\sin(3x - 4x) + \sin(3x + 4x)) dx \]

\[ = \frac{1}{2} \int (- \sin(x) + \sin(7x)) dx = \frac{1}{2} (\cos(x) - \frac{\cos(7x)}{7}) + C \]

5. Other Trigonometric Integrals

Most other trigonometric functions are not so rigid, and often completely different techniques may be required for other types. We finish by looking at a non standard example.

Example 5.1.

\[ \int \cos^2(x) \tan^2(x) dx = \int \cos^2(x) \frac{\sin^2(x)}{\cos^2(x)} dx \]

\[ = \int \sin^2(x) dx = \int \frac{1}{2} (1 - \cos(2x)) dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C \]