Remark: In the usual case where $\mu = 1$, the new algorithm can be written as

$$\Delta p_{n+1} = \frac{b_r q_n - 2\Delta p_n}{n+1} + \Delta p_n$$  \hspace{1cm} (8)$$

$$q_{n+1} = \frac{p_n - q_{n-2} + q_{n-1}}{n+1}$$  \hspace{1cm} (9)$$

where $q_{n+1} = \Delta p_{n+1} + p_n$ and on the understanding that $q_{-1} = 0$.

Illustrative example: Consider a URC line, with total resistance $R$ and capacitance $C$, terminated with parallel resistor ($R_L = R$) and capacitor ($C_L = C$). For convenience, normalizing RC to unity, the voltage-to-voltage transfer function of the voltage-driven loaded RC line may be expressed as [4]

$$\tilde{g}(s) = \frac{1}{\cosh(\sqrt{s}) + (\frac{1}{s} + 1) \sinh(\sqrt{s})}$$

Denoting the unit step function by $U(t)$, let $f(t) = r(t) - (1/2)U(t)$ be the transient part of the step response $r(t)$ and let $L_n$ be the Laguerre polynomial of degree $n$. The values of $r_n$ which yield the best Laguerre approximation to $f(t)$ of the form

$$\tilde{f}(t) = e^{-\alpha t} \sum_{n=0}^{N-1} (r_n - u_n/2)L_n(\beta t) \quad t \geq 0$$

in the sense of minimising the so-called integral squared error (ISE)

$$Q = \int_0^\infty e^{-(\alpha-2\beta)t} \left[ \tilde{f}(t) - f(t) \right]^2 dt$$

are given by the $\tau$-transform $R(\tau) = \beta e^{(\tau-1)} \hat{R}(\tau)$ where $\hat{R}(\tau) = \hat{g}(\tau)/\beta(\tau-1)$, while $u_n = \beta^\alpha(\tau-1)\hat{g}(\tau)$ [10, 11]. Letting $\beta = 2\alpha = 9$ (the choice is not critical) yields $R(\tau) = \beta e^{(\tau-1)/(2-2\tau^-1)}$ where $D(\tau) = (1-\tau^-1)P(\tau) + (2.5 + 3\tau^-1 + 0.5\tau^-2)Q(\tau)$ with $P$, $Q$ defined by eqns. 1 - 3, $\alpha = 1$, $\alpha = 4.5$ and $\mu = 1$. In view of its expression, $D(\tau)$ may be regarded as the $\tau$-transform of $d_0 = p_0 - p_2 = 2.5q_2 + 3q_3 + 3q_4 + 0.5q_5$. This relationship permits computation of the first $N$ terms of the sequence ($d_n$) from only the first $N$ terms of each of the sequences ($r_n$) and ($u_n$) computed via the proposed algorithm. Finally, long division of $(2-2\tau^-1)$ by $\sum_{n=0}^{N-1} d_n \tau^n$ yields the first $N$ coefficients $r_n$.

Using eqns. 4, 8 and 9, we obtain $p_0 = 4.231, q_0 = 1.938, p_1 = 8.721, q_1 = 2.393, p_2 = 13.880, q_2 = 3.214, ...$; thus

$$D(z) = 0.076 + 20.268z^{-1} + 25.532z^{-2} + ...$$

and because $u_n/2 = (-1)^n$ the third-order approximation is obtained as

$$\tilde{f}(t) = e^{-\frac{\alpha}{2}t} \left[ -0.780L_0(0t) + 0.223L_1(0t) - 0.029L_2(0t) \right]$$

and finally $\tilde{f}(t) = \tilde{f}(t) + \frac{1}{2} U(t)$ is the approximate step response.

Conclusion: It is felt that the simplicity of the new proposed algorithm increases the attraction for using the Laguerre transform in the signal analysis of transients on uniformly distributed RCG lines, a technique which does not require any initial approximation and can theoretically give very accurate time responses because the Laguerre functions are complete in $L^2[0, \infty]$. In practice, a few terms in the Laguerre series expansion are generally sufficient; an improvement in accuracy is obtained by simply adding further terms in the truncated Laguerre series and usually, for a given number of terms, the quadratic error can be reduced further by adjusting parameters $\alpha$ and $\beta$. The method is applicable to networks made up of lumped and distributed elements intermixed. Moreover, equality constraints can be set to preserve some properties of the original network [12], such as for example $\tilde{F}(0) = n(0)$.

References


Special singularity integral identity and its applications in electrical circuits

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A new special singularity integral identity involving the product of the impulse symbol and the unit step function is proposed. A proof of this integral identity is provided followed by its application in the solution of a simple well-known ideal example problem in electrical circuits.

Introduction: A new technique is proposed for calculating electromagnetic force or energy in idealised problems involving perfect conductors and electrical circuits. In this technique, relevant physical parameters such as voltage, current, charge, electric field, etc. are expressed in terms of either the impulse symbol singularity $\delta(t)$ or the unit step function singularity $u(t)$. As a result, interesting physical quantities such as electromagnetic force and energy can be expressed directly in terms of a new special singularity integral identity (i.e. the sifting-integral of $u(t)$ [1]) given by:

$$\int_{-\infty}^{\infty} \delta(t) u(t) dt = \frac{1}{2}$$

Using this new formulation based on eqn. 1, electromagnetic force and energy can be directly and straightforwardly solved in analytical closed form. In this Letter we first present a proof of this new special singularity integral identity (eqn. 1) and then demonstrate its utility by applying it to solve a well-known idealised electrical circuit example problem.
Proof of eqn. 1: Following Bracewell [1], since the impulse symbol singularity is not a proper function (i.e., its value at \( t = 0 \) is not defined) but is rather a generalised function, the result of any integral such as eqn. 1 which includes the impulse symbol must be interpreted by replacing the impulse by an appropriately chosen sequence of functions which in the limit exhibit impulsive behaviour. Consider Fig. 1, where we use the unit step function singularity \( \delta(t) \) and replace the impulse symbol singularity \( \delta(t) \) with the rectangular pulse function \( \Pi(t/\tau) \) in order to prove the validity of eqn. 1. As shown in Fig. 1, by performing the multiplication and integration of these two functions and then taking the limit \( \tau \to 0 \), we obtain

\[
\lim_{\tau \to \infty} \int_{-\infty}^{\infty} \tau^{-1} \Pi(t/\tau) u(t) \, dt = \int_{-\infty}^{\infty} \delta(t) u(t) \, dt \\
= \int_{0}^{\infty} \delta(t) u(t) \, dt = \frac{u(0^+) + u(0^-)}{2} = \frac{1}{2}
\]  

(2)

Therefore, the eqn. 1 identity has been proved. Next, we consider a simple well-known idealised electrical circuit example problem where we use the special singularity integral identity (eqn. 1) to calculate the relevant energies in the system.

Example-charging RC circuit: Consider the RC circuit shown in Fig. 2. It is well-known from basic circuit theory that the resulting voltage and the current of the charging capacitor are given by \( v_C(t) = v_C [1 - e^{-t/RC}] \) and \( i_C(t) = (v_C/R) e^{-t/RC} \), respectively, from which the energy \( W_{\text{source}} \) supplied by the DC voltage source, the energy \( W_S \) liberated by the switch S into free space as electromagnetic radiation, the energy \( W_R \) dissipated in the resistor \( R \), and the energy \( W_C \) stored in the capacitor \( C \) can be calculated as

\[
W_{\text{source}} = \int_{-\infty}^{\infty} V_0 i_C(t) \, dt = CV_0^2
\]  

(3)

\[
W_S = \int_{-\infty}^{\infty} V_0 \left[ 1 - u(t) \right] i_C(t) \, dt = 0
\]  

(4)

\[
W_R = \int_{-\infty}^{\infty} R i_C(t) i_C(t) \, dt = \frac{1}{2} CV_0^2
\]  

(5)

\[
W_C = \int_{-\infty}^{\infty} V_C(t) i_C(t) \, dt = \frac{1}{2} CV_0^2
\]  

(6)

Note that the conservation of energy principle is satisfied (i.e., \( W_{\text{source}} = W_S + W_R + W_C \)). In addition, it is observed that the energy values are independent of the resistor value \( R \) and therefore the value of \( R \) only affects the time constant \( \tau = RC \) of the circuit. We now consider the same circuit problem shown in Fig. 2 with \( R = 0 \). In this idealised case, the capacitor voltage and current can now be expressed in terms of the unit step function and the impulse symbol singularities as \( v_C(t) = V_0 u(t) \) and \( i_C(t) = CV_0 \delta(t) \), respectively [1]. Then, using the special singularity integral identity (eqn. 1), the energy calculations for the \( R = 0 \) case directly follow:

\[
W_{\text{source}} = \int_{-\infty}^{\infty} V_0 CV_0 \delta(t) \, dt = CV_0^2
\]  

(7)

\[
W_S = \int_{-\infty}^{\infty} V_0 [1 - u(t)] CV_0 \delta(t) \, dt = \frac{1}{2} CV_0^2
\]  

(8)

\[
W_R = \int_{-\infty}^{\infty} \left[ \frac{RI_C(t)}{V_0} \right] i_C(t) \, dt \bigg|_{t=0} = 0
\]  

(9)

\[
W_C = \int_{-\infty}^{\infty} V_C(t) i_C(t) \, dt = \frac{1}{2} CV_0^2
\]  

(10)

These results are as expected (i.e., \( W_{\text{source}} = W_S + W_R + W_C \)). However, it is interesting to observe that in the limiting case when \( R = 0 \), half of the energy supplied by the DC voltage source is liberated as electromagnetic radiation, as heat, light and sound generated by a spark discharge at the switch, or as heat within the voltage source. All of the energy supplied is dissipated in the resistor when \( R \neq 0 \), unless \( R \) is so small that the peak current launches noticeable radiation.

Conclusion: The literature available on this subject strongly advocates that the sifting-integral of \( \delta(t) \) cannot be given a consistent meaning and must be avoided [2] and therefore, by implication, the special singularity integral identity (eqn. 1) is undefined. However, in light of the proof provided in this paper for eqn. 1 followed by its direct application in a simple well-known idealised electrical circuit example problem, we firmly conclude that the special integral identity given by eqn. 1 is both valid and relevant in obtaining the solutions of many interesting problems in physics and engineering and therefore deserves to be included and emphasised in the scientific literature.

References